Review of spectral analysis methods applied to sea level anomaly signals

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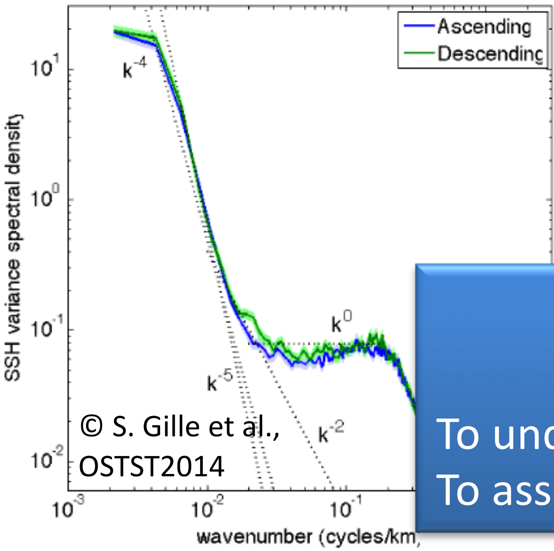
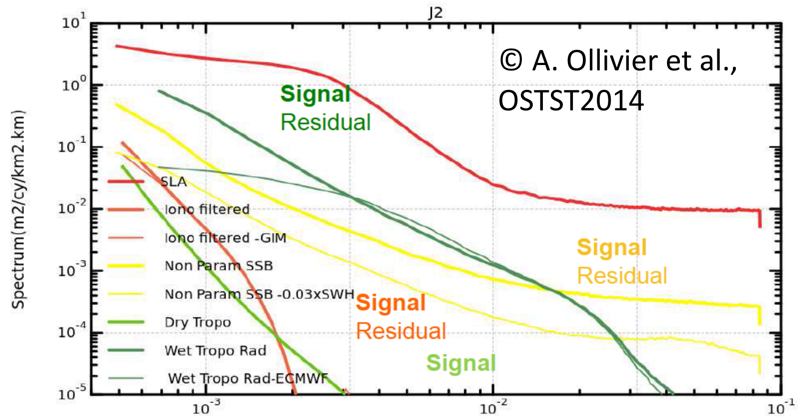
Context of the presentation





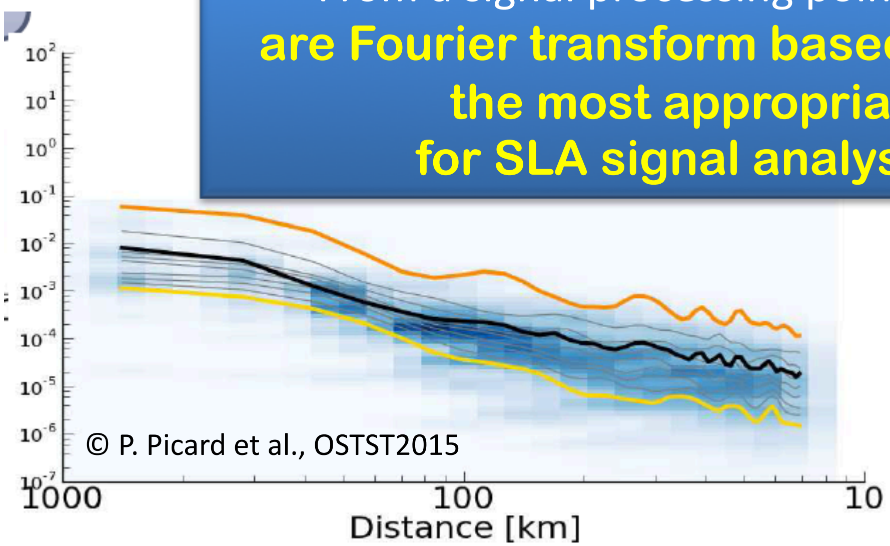
SPECTRAL ANALYSIS of sea level anomalies (SLA) widely used in the altimetry community:

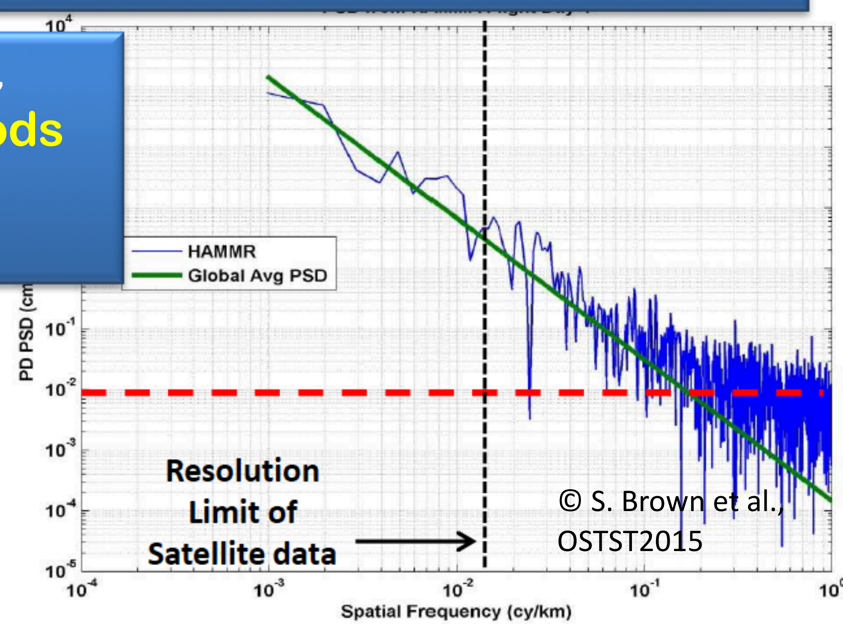
To understand the geophysical content of measured signals,

To assess and compare the performance of missions

From a signal processing point of view,

are Fourier transform based methods the most appropriate for SLA signal analysis?



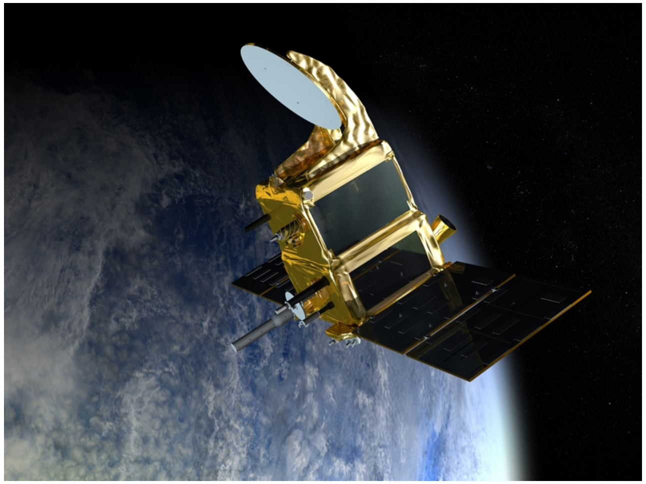


Outline of the talk

Study funded by CNES

Review of spectral analysis methods

1. What is spectral analysis?
2. The Welch periodogram
   a. Influence of the weighting temporal window
   b. Influence of the length and number of segments
   c. How to better estimate the slope?
3. Other methods of spectral analysis?



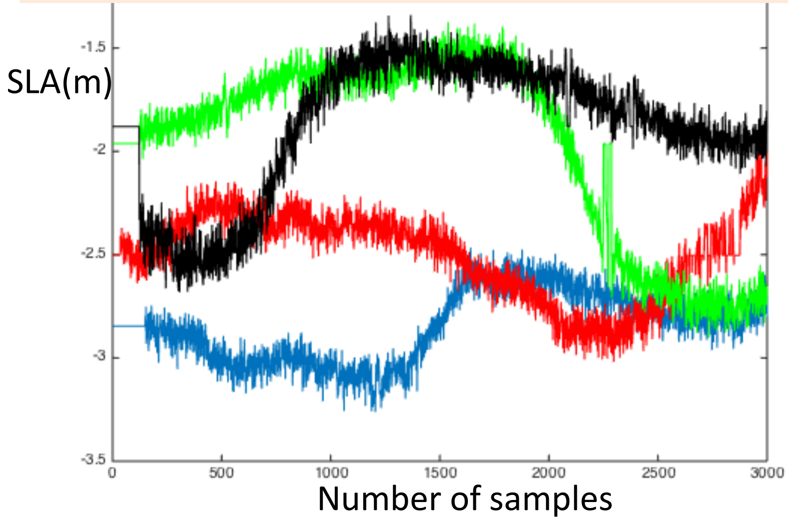
Comparisons made
on simulated Sea Level Anomalies (SLA)
and
on real signals from SARAL/AltiKa, Agulhas current area

1. What is Spectral Analysis?

From a theoretical point of view

Observed signals = realisations of a stochastic process

Agulhas Current
Sea Level Anomaly (SLA) measurements



Power Spectrum Density
(PSD or « spectrum »)

$$S_x(f) = \lim_{L\to\infty} E[\frac{1}{L}|X_L(f)|^2]$$

$$X_L(f) = \mathrm{FT}\{x(t), t = 0, ..., L\}$$

To compute a PSD, one needs to:

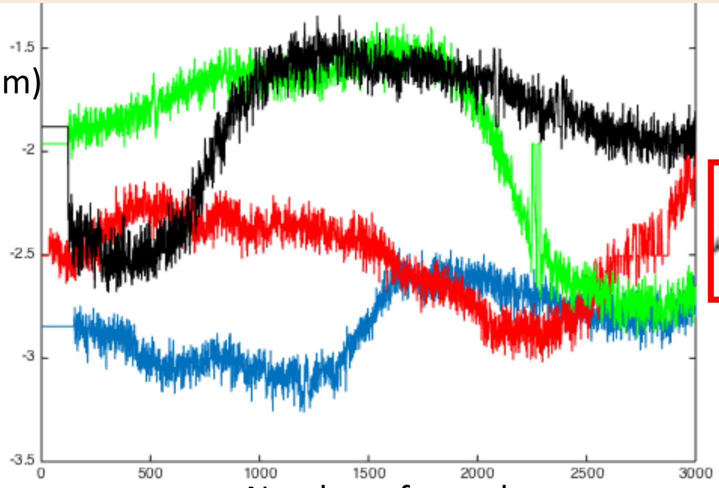
- **Know the process on a finite temporal window L,**
- **Compute the squared modulus of the Fourier transform**
- **Compute the *mathematical expectation* (statistics?)**
- **Compute the *limit when L tends to infinity* (how?)**

1. What is Spectral Analysis?

From a practical point of view

Observed signals = realisations of a stochastic process

Agulhas Current
Sea Level Anomaly (SLA) measurements



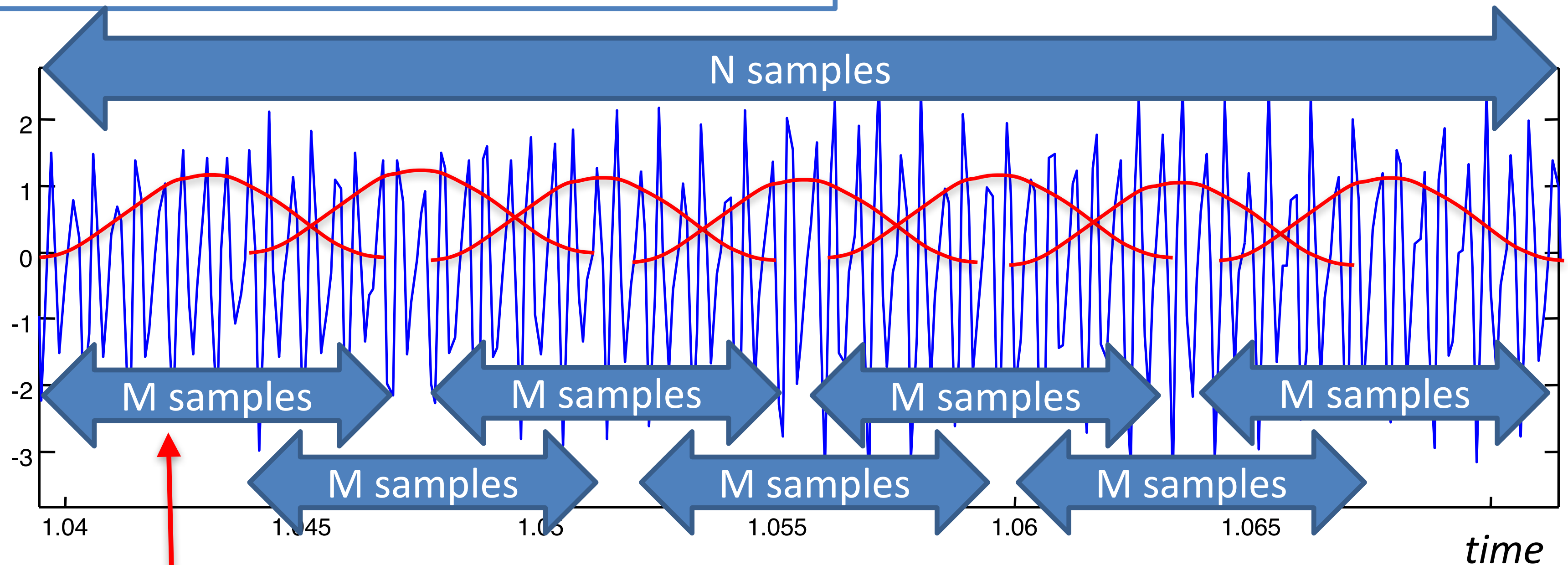
PSD Estimation
= Periodogram

$$\hat{S}_{Per}(f) = \frac{1}{N} |\ DFT\ \{x(n), n = 0, ..., N-1\}|^2$$

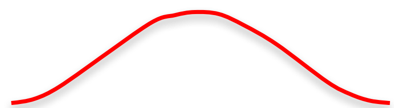
It looks like the PSD but
No stochastic process,
No mathematical expectation,
No limit computation

Periodogram =
One possible estimator, but with bias and variance

2. The Welch periodogram



$$\hat{S}_{PerMod}(f) = \frac{1}{M}|TFD(x(n)w(n))|^2$$



Weighting temporal window
Which one?

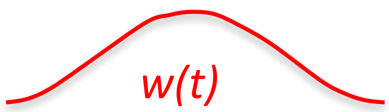
$$\hat{S}_{Welch}(f) = \frac{1}{L}\sum_{k=1}^{L}\hat{S}_{PerMod_k}(f)$$

Number of segments?
Size?

2. The Welch periodogram

a. Influence of the weighting temporal window

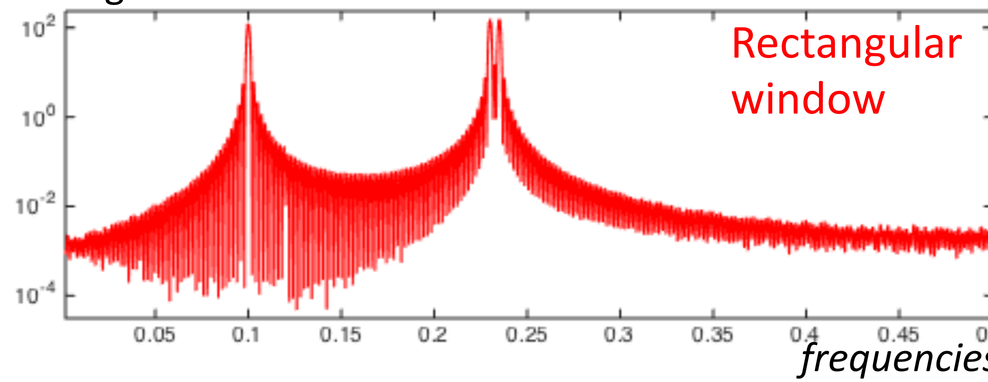
Which weighting temporal window? Depends what you are looking for…

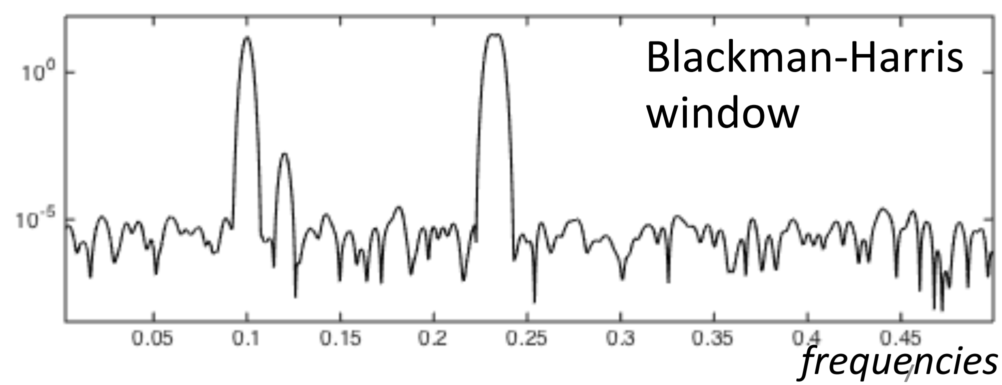


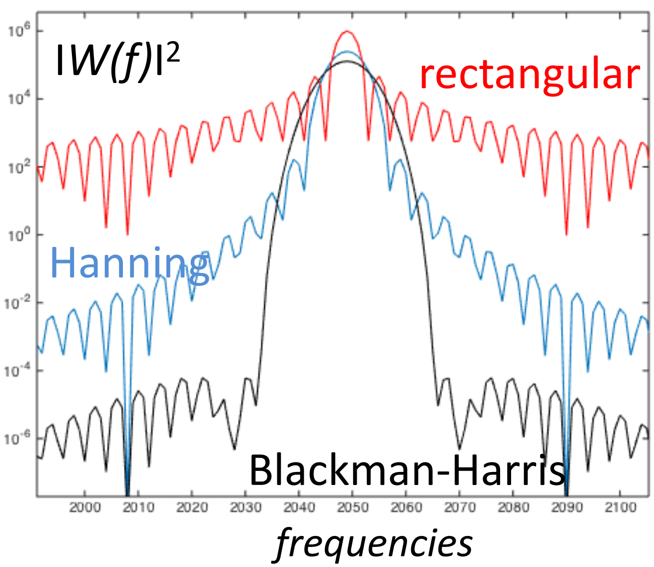
Convolutive bias

$$\mathrm{E}[\hat{S}_{PerMod}(f)] = S(f) * |W(f)|^2$$

Illustration on an academic example:
Signal = 4 sinusoids + white Gaussian noise



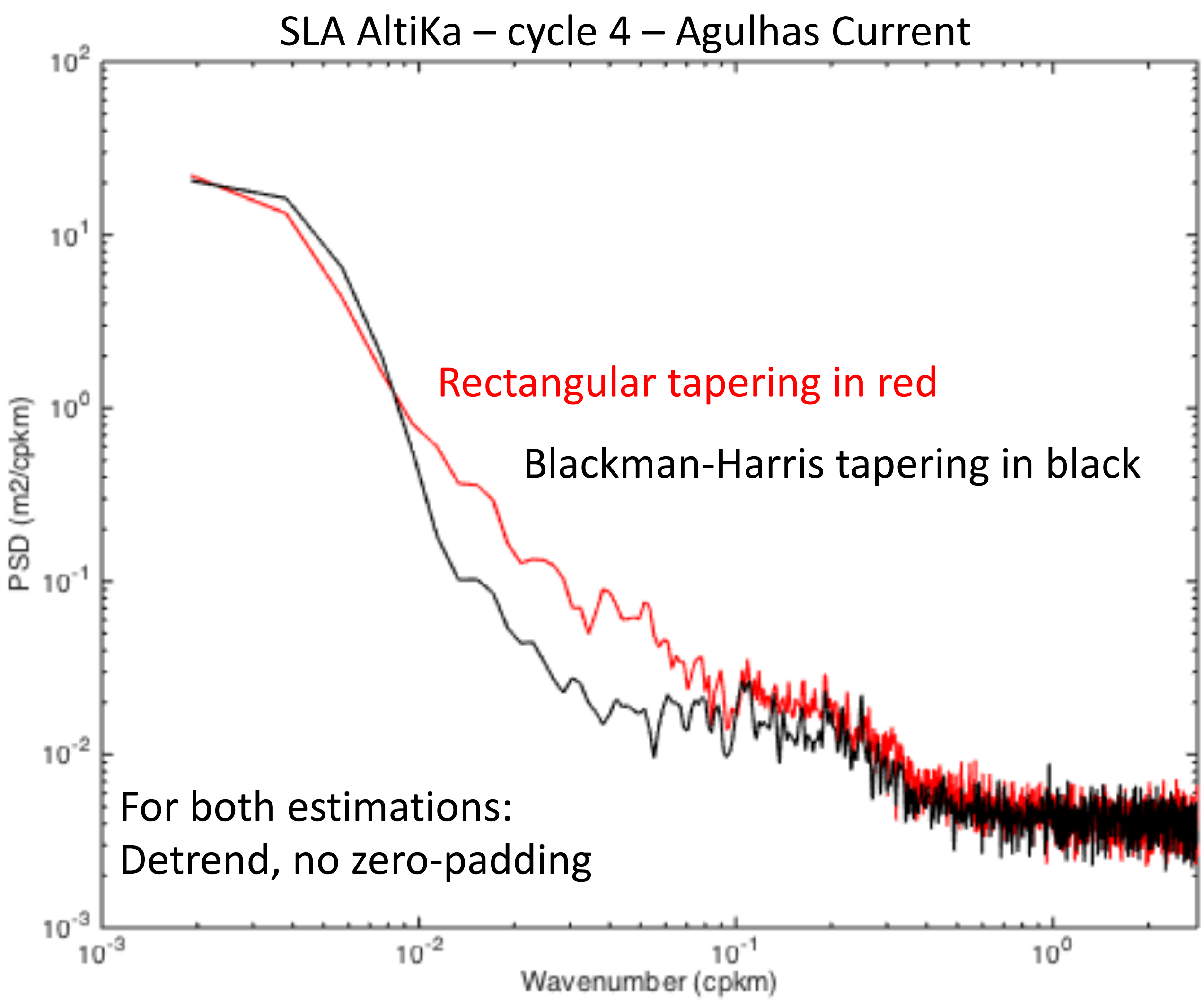




Frederic J. Harris,
On the use of Windows for Harmonic Analysis with the Discrete Fourier Transform,
Proceedings of the IEEE, Vol.66, No.1, January 1978, pp 51–83.

2. The Welch periodogram

a. Influence of the weighting temporal window



Two
PSD estimations,
both biased
(convolutive bias)

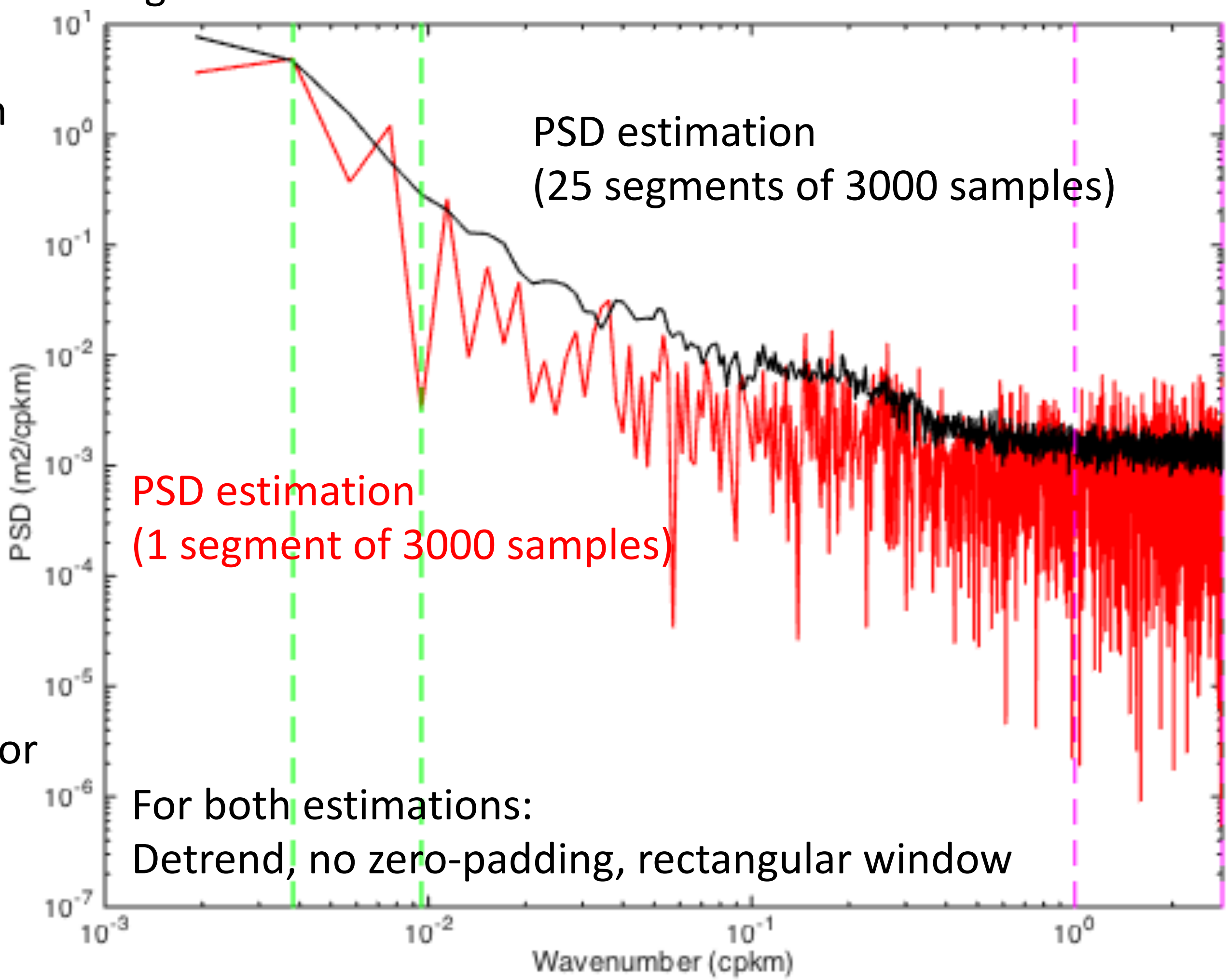
2. The Welch periodogram

b. Influence of the length & number of segments

SLA AltiKa – cycle 4
Agulhas Current

Length of segments
= Frequency resolution

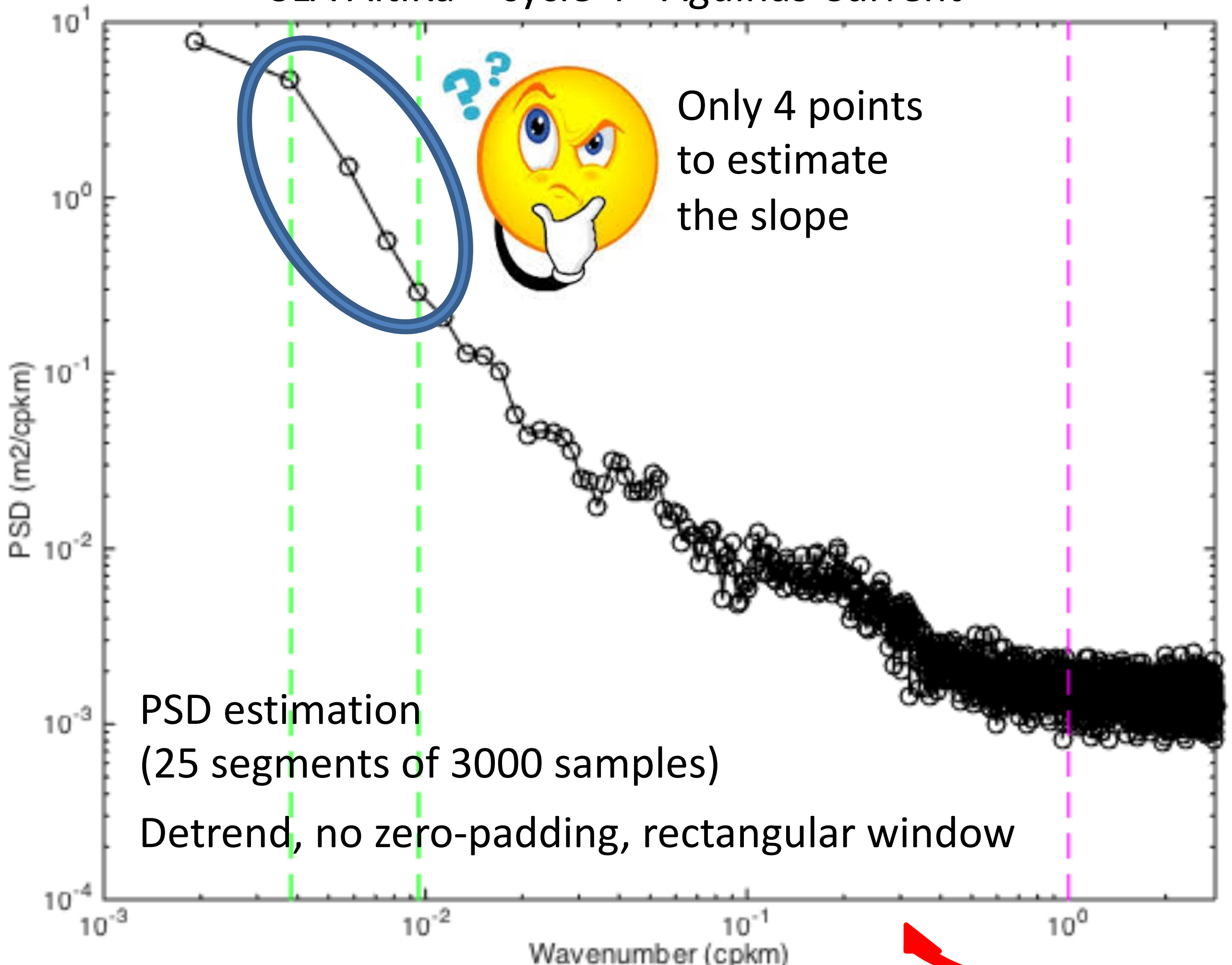
Number of segments
= Variance
of the spectral estimator



2. The Welch periodogram

c. How to better estimate the slope?

SLA AltiKa – cycle 4 - Agulhas Current



It comes from the frequency scale of the DFT:

Classical = N frequency values uniformly distributed between 0 and F_s

BUT
In altimetry

Frequency scale = logarithmic

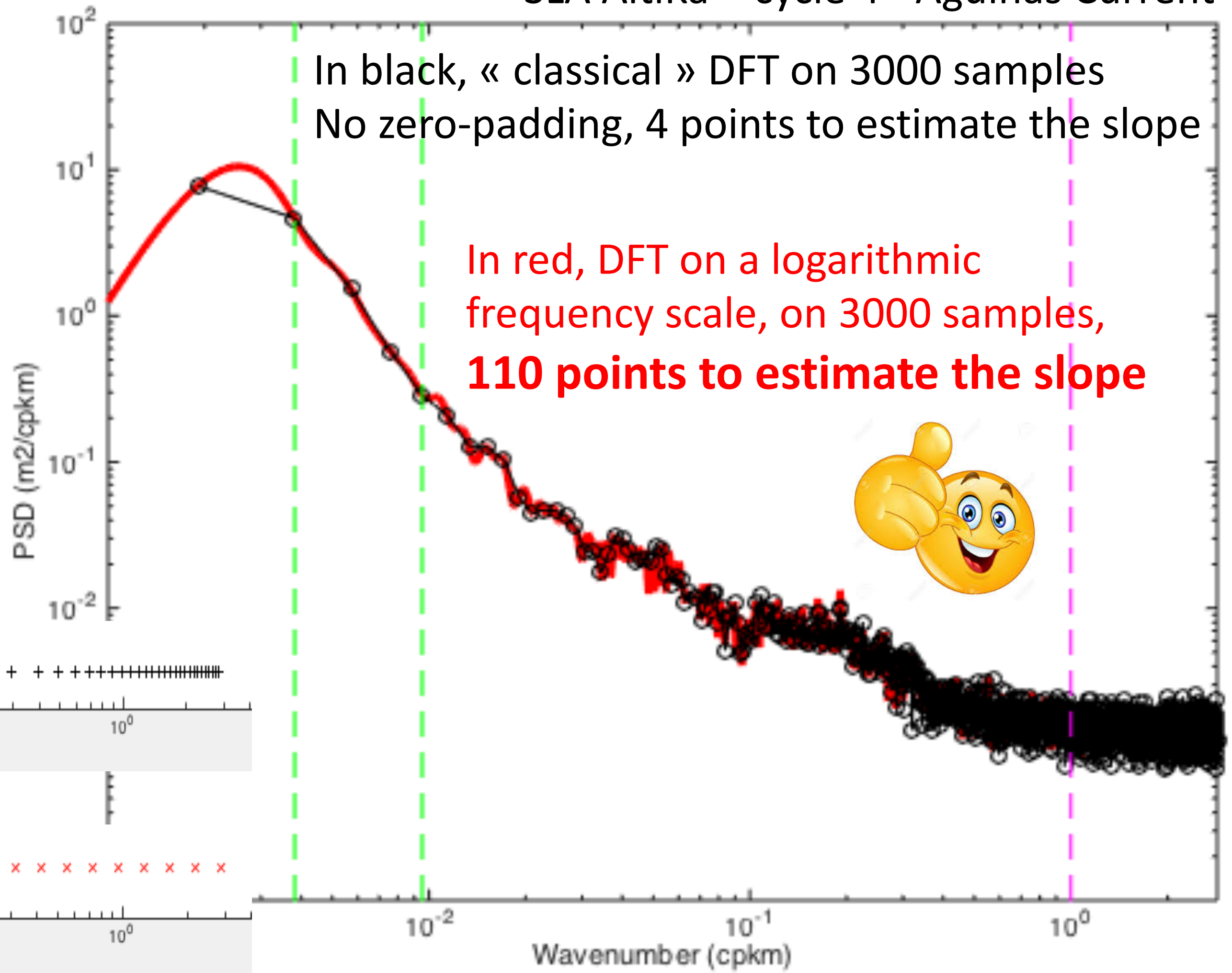
2. The Welch periodogram

c. How to better estimate the slope?

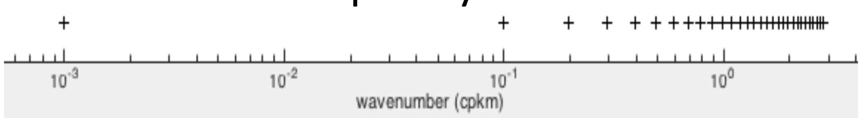
Well-suited to SLA spectra:

DFT computed on a **logarithmic frequency scale**

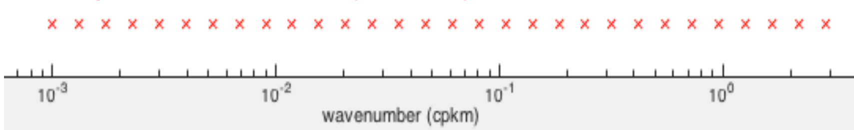
SLA AltiKa – cycle 4 - Agulhas Current



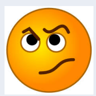
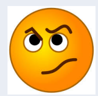
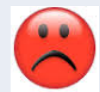
« classical » frequency scale



Logarithmic frequency scale



2. The Welch periodogram

Spectral analysis	Bias	Window choice	Variance	Fast algo	log frequency scale
Fourier-based methods					

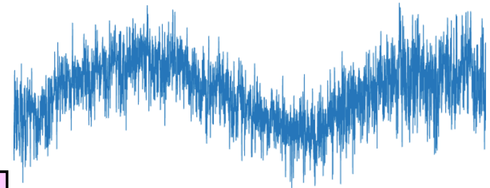
May induce bias
In slope estimation

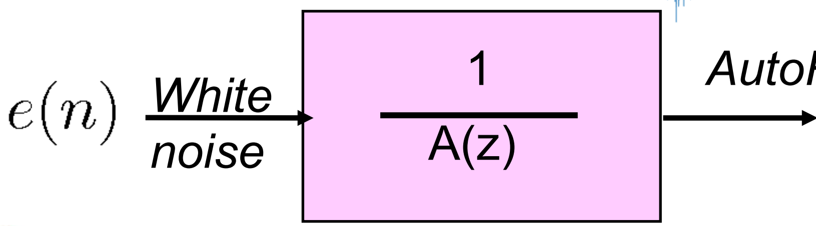
Necessary to
average
several segments

Other methods of spectral analysis?

3. Other methods of spectral analysis?

Parametric spectral analysis





AutoRegressive (AR) modeling

$$x(n) = -\sum_{k=1}^{p} a_k x(n-k) + e(n)$$

AR spectral estimator

$$S_{AR}(f) = \frac{\sigma_e^2}{\left|1 + \sum_{k=1}^{p} a_k e^{i2\pi f k}\right|^2}$$

Not suffering from effects of windowing
Better stability for short signal segments
Better spectral resolution

Choice of order *p*
Not reversible
Slightly more complicated to code
parametric model – has to be adapted to signals of interest

3. Other methods of spectral analysis?
Parametric spectral analysis

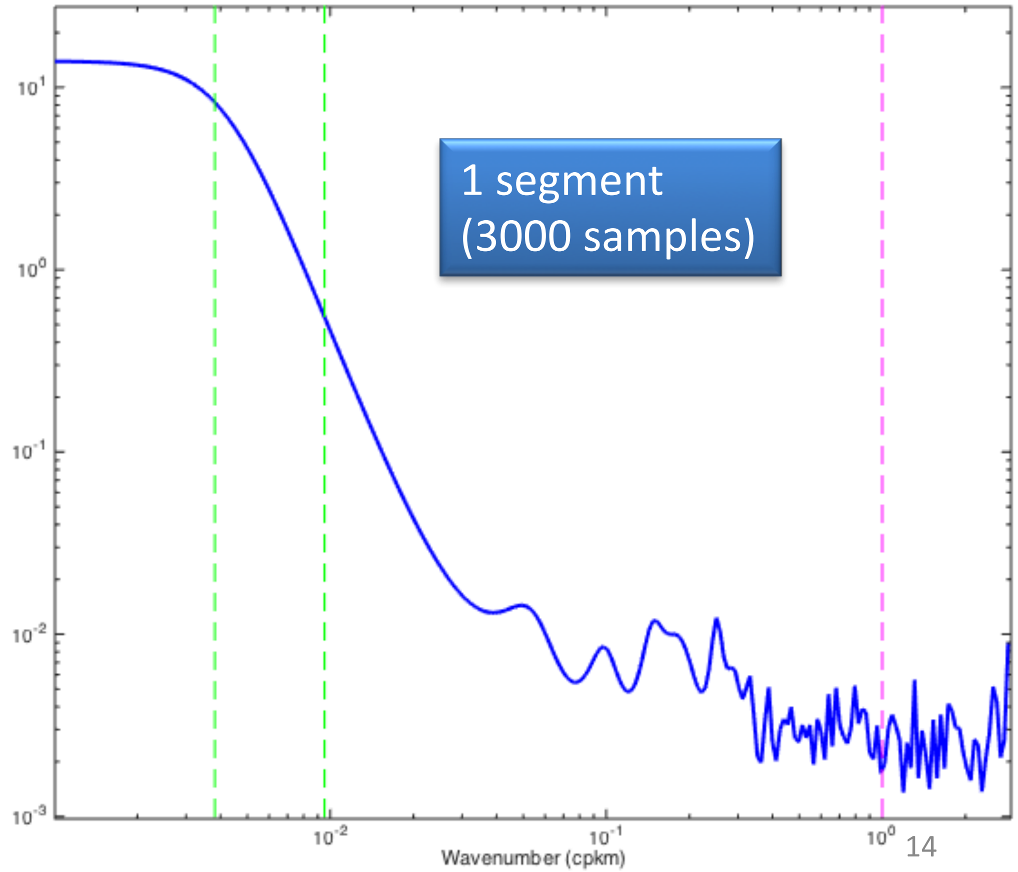
AutoRegressive Spectral Analysis (AR)

Spectral estimation possible
on small segments

Slope can be estimated

No need to average

SLA AltiKa – cycle 4 – Agulhas Current



4. Conclusions

Study funded by CNES

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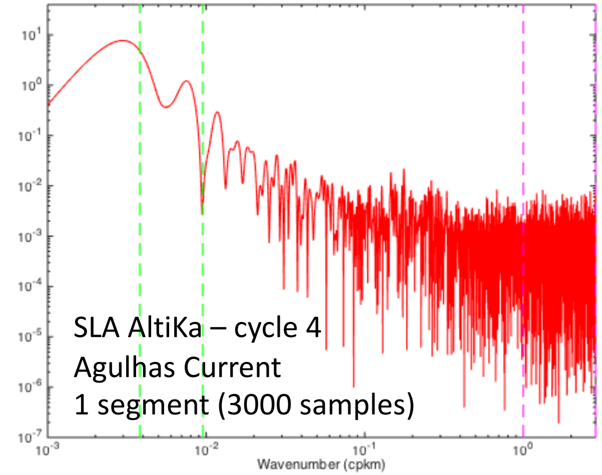
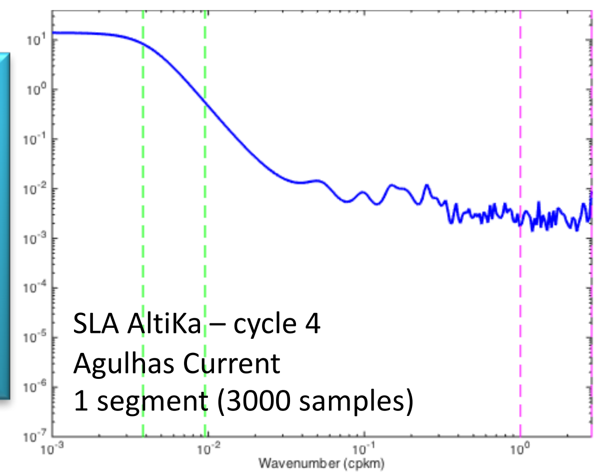
Spectral analysis based on Fourier transform

- Zero-padding or logarithmic frequency scale good for slope estimation
- Bias due to any weighting temporal window
- Large variance => necessary to average several segments



AR spectral estimation

- No windowing effect,
- Low variance, no need to average several segments
- Logarithmic frequency scale available
- Choice of order *p*

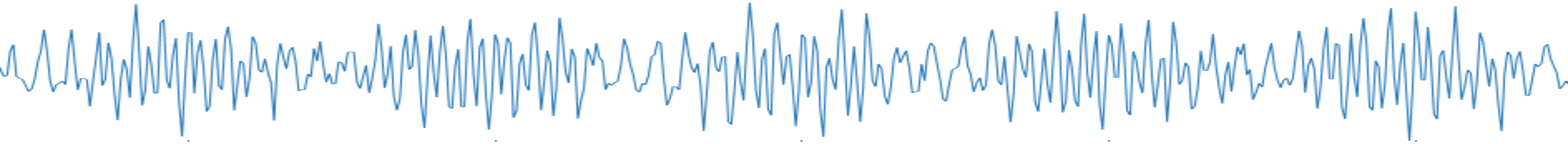


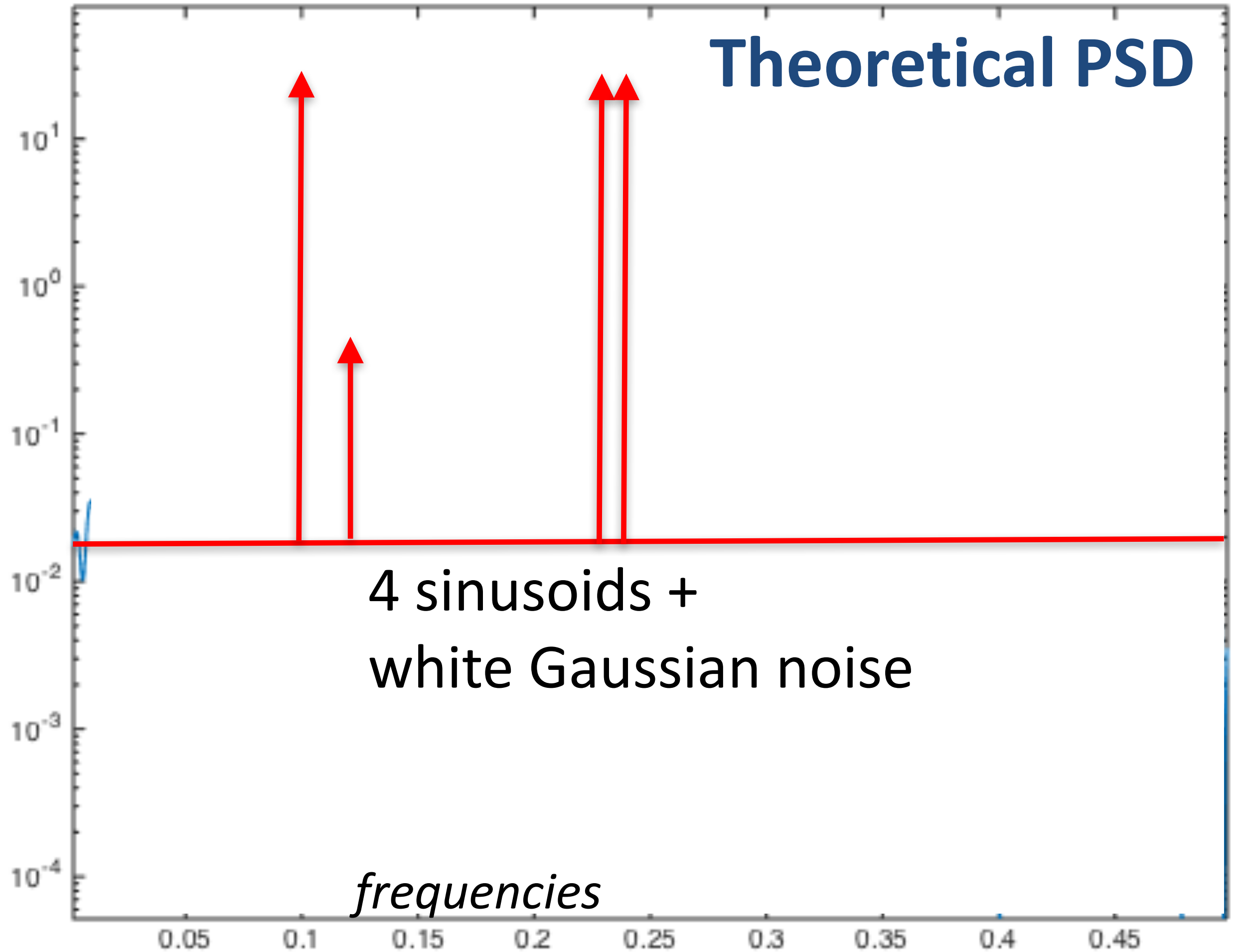
To be further investigated on Doppler altimetry signals

Thank you for your attention

Supplementary slides

To illustrate bias and variance of Fourier based methods:
An « academic » example





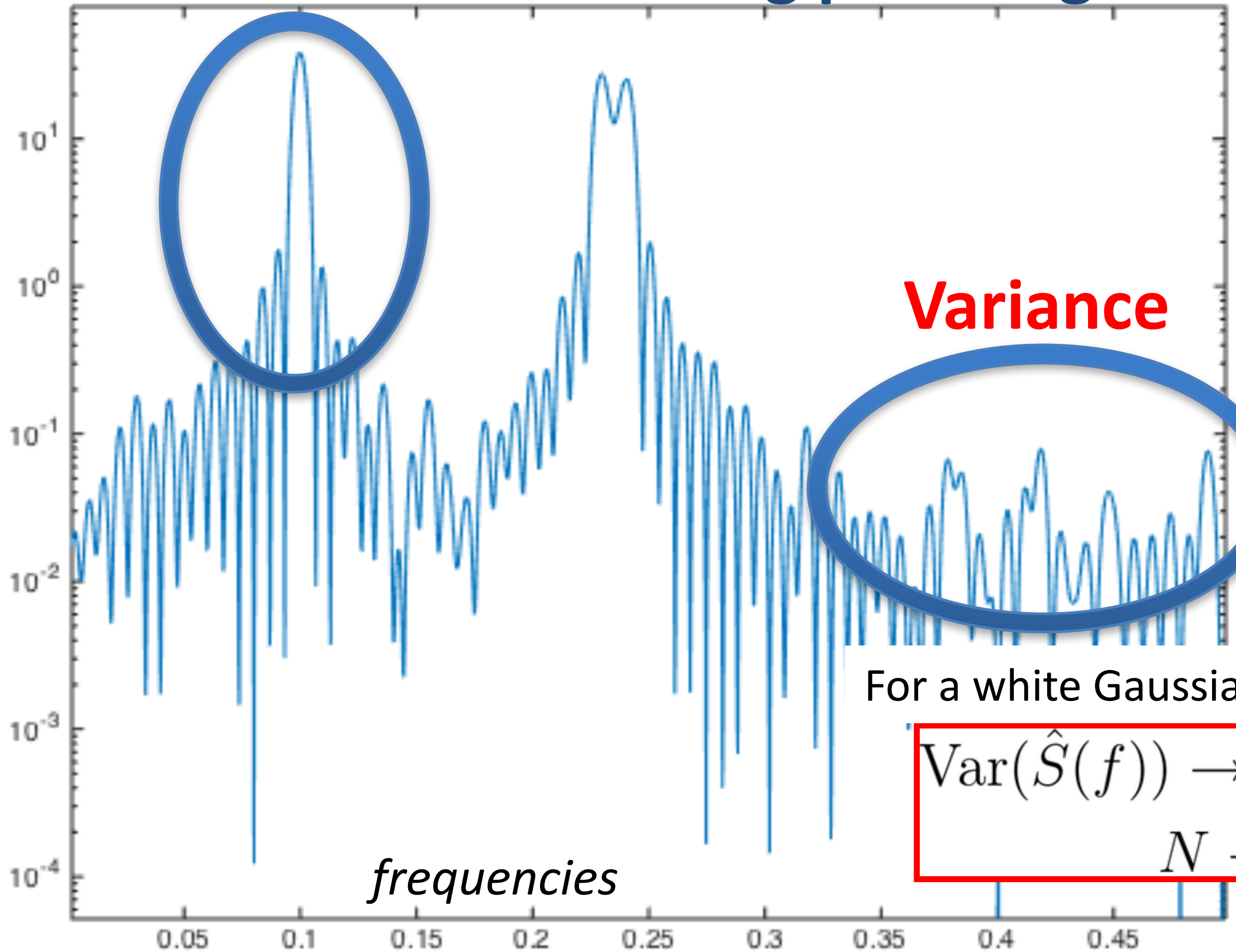
Supplementary slides

To illustrate bias and variance of Fourier based methods:
An « academic » example

PSD estimation using periodogram

Convolutive bias

Variance

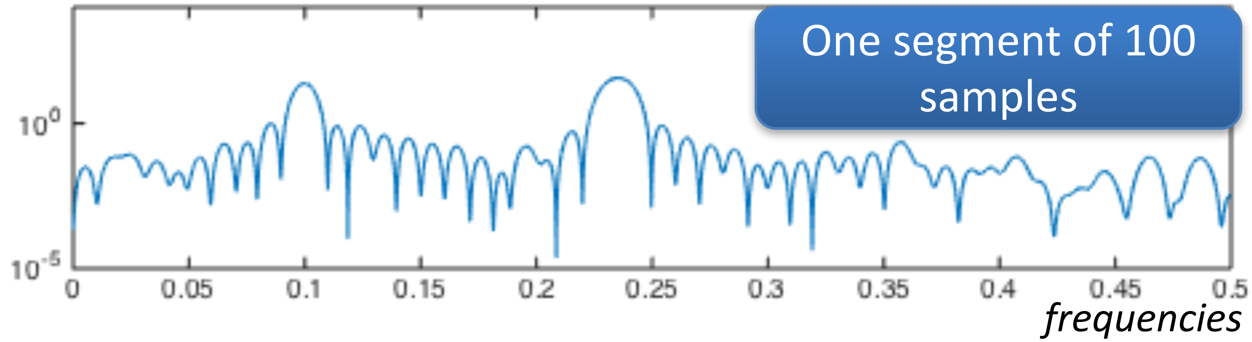


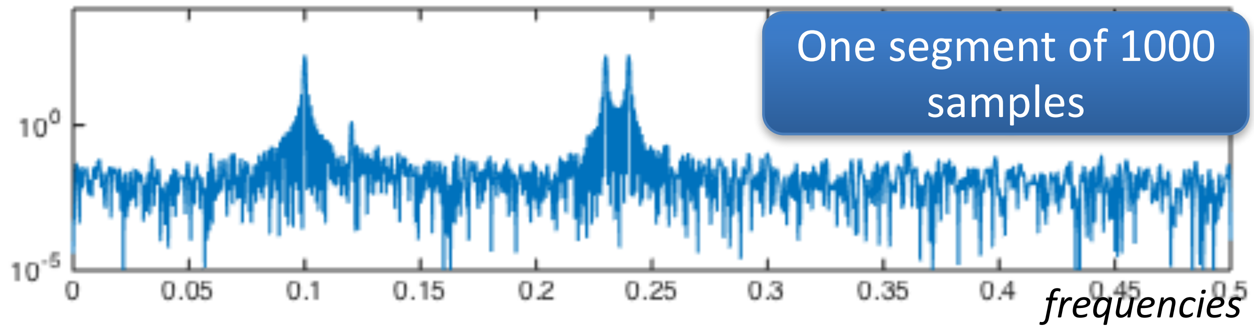
For a white Gaussian noise

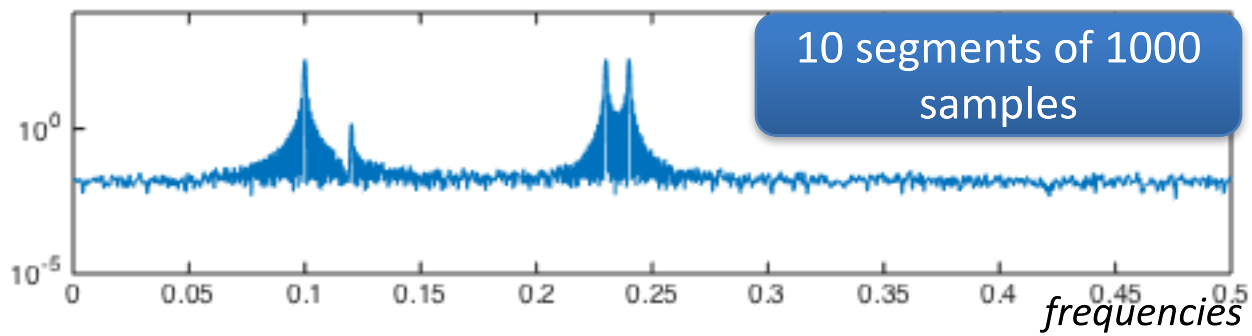
$$\mathrm{Var}(\hat{S}(f)) \longrightarrow S^2(f)$$
$$N \longrightarrow +\infty$$

Supplementary slides

To illustrate the influence of the length and the number of segments in Fourier based methods:
An « academic » example







Length of segments
= Frequency resolution

Number of segments
= Variance
of the spectral estimator

Supplementary slides

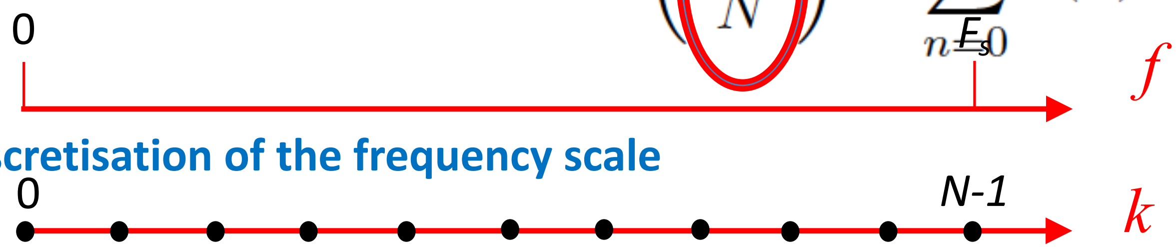
To illustrate what is ZERO-PADDING (interesting to better estimate the slope)

From the Discrete Fourier transform …

$$\text{For } 0 \le f \le F_s,\ X_D(f) = \sum_{n=0}^{N-1} x(n) e^{-i2\pi f n T_s}$$

… To the algorithm

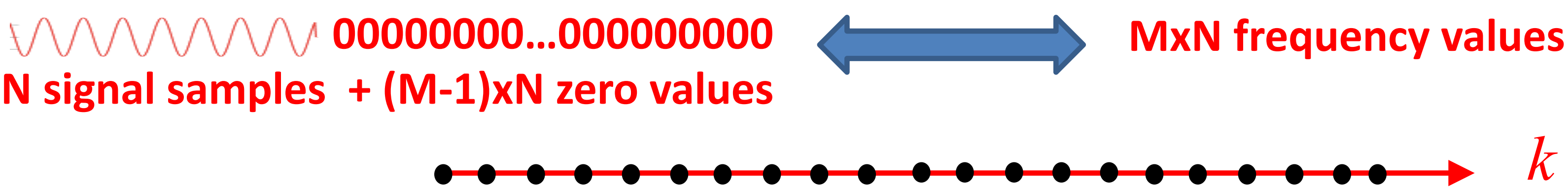
$$\text{For } k = 0, ..., N-1,\ X_D\left(\frac{kF_s}{N}\right) = \sum_{n=0}^{N-1} x(n) e^{-i2\pi \frac{kF_s}{N} n T_s}$$



Necessary discretisation of the frequency scale

N signal samples ⟷ **N frequency values**

… ZERO-PADDING: just add zero values after the signal samples (same as window effect)



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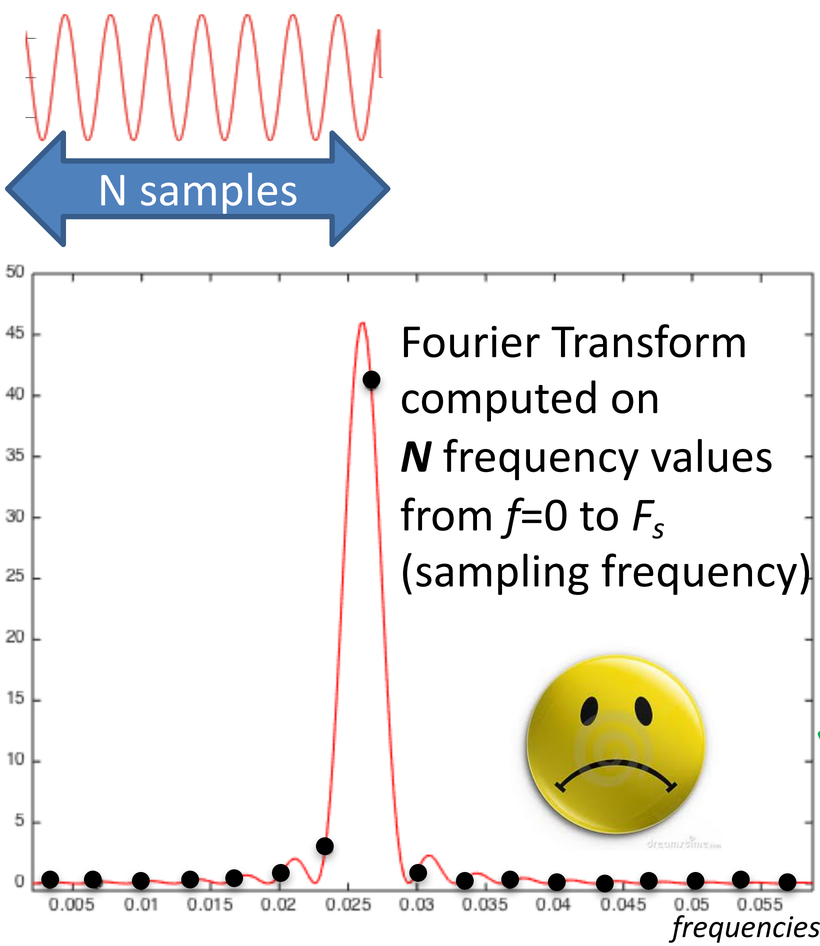
N signal samples + (M-1)xN zero values ⟷ **MxN frequency values**

A finer discretisation of the frequency scale, a better representation

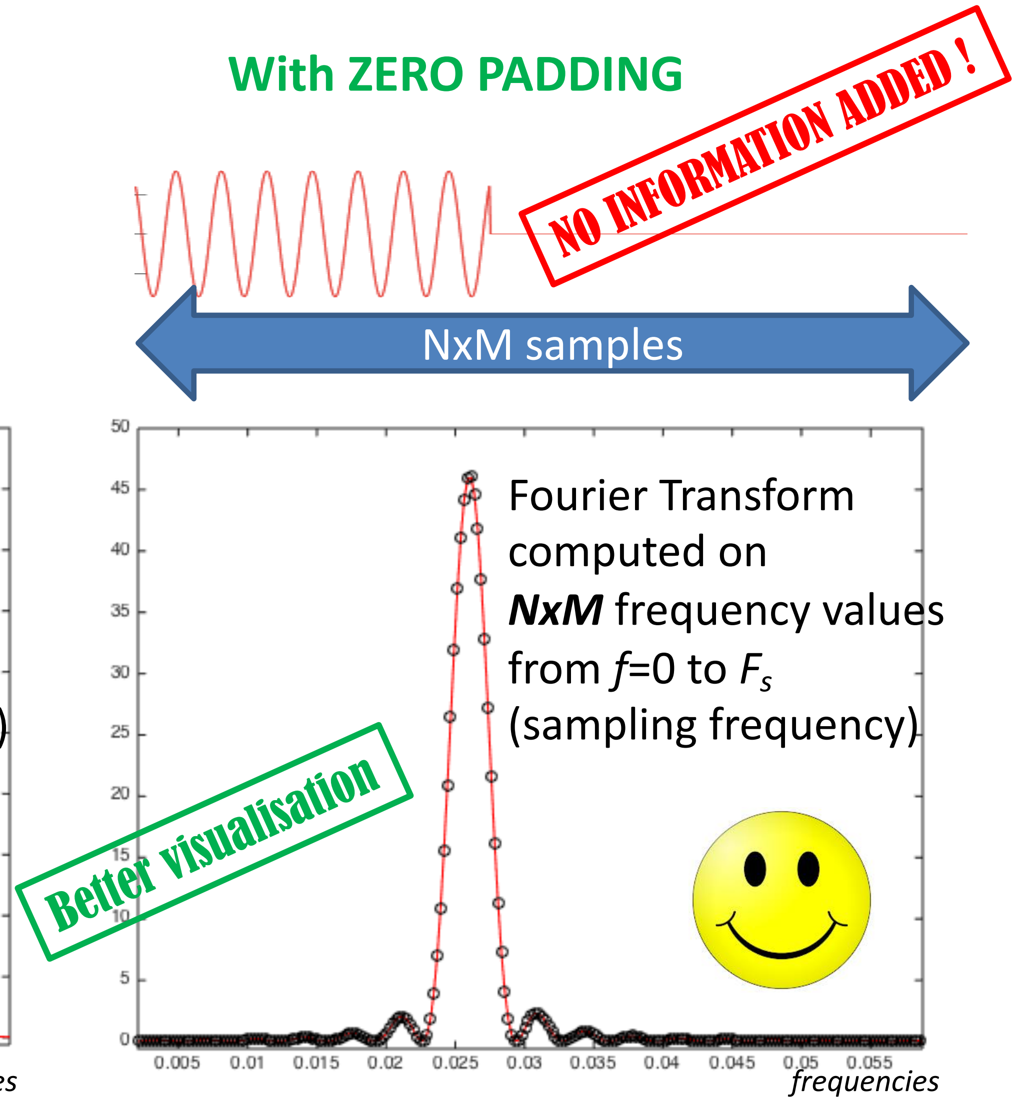
Supplementary slides

To illustrate what is ZERO-PADDING (interesting to better estimate the slope)

Without ZERO PADDING



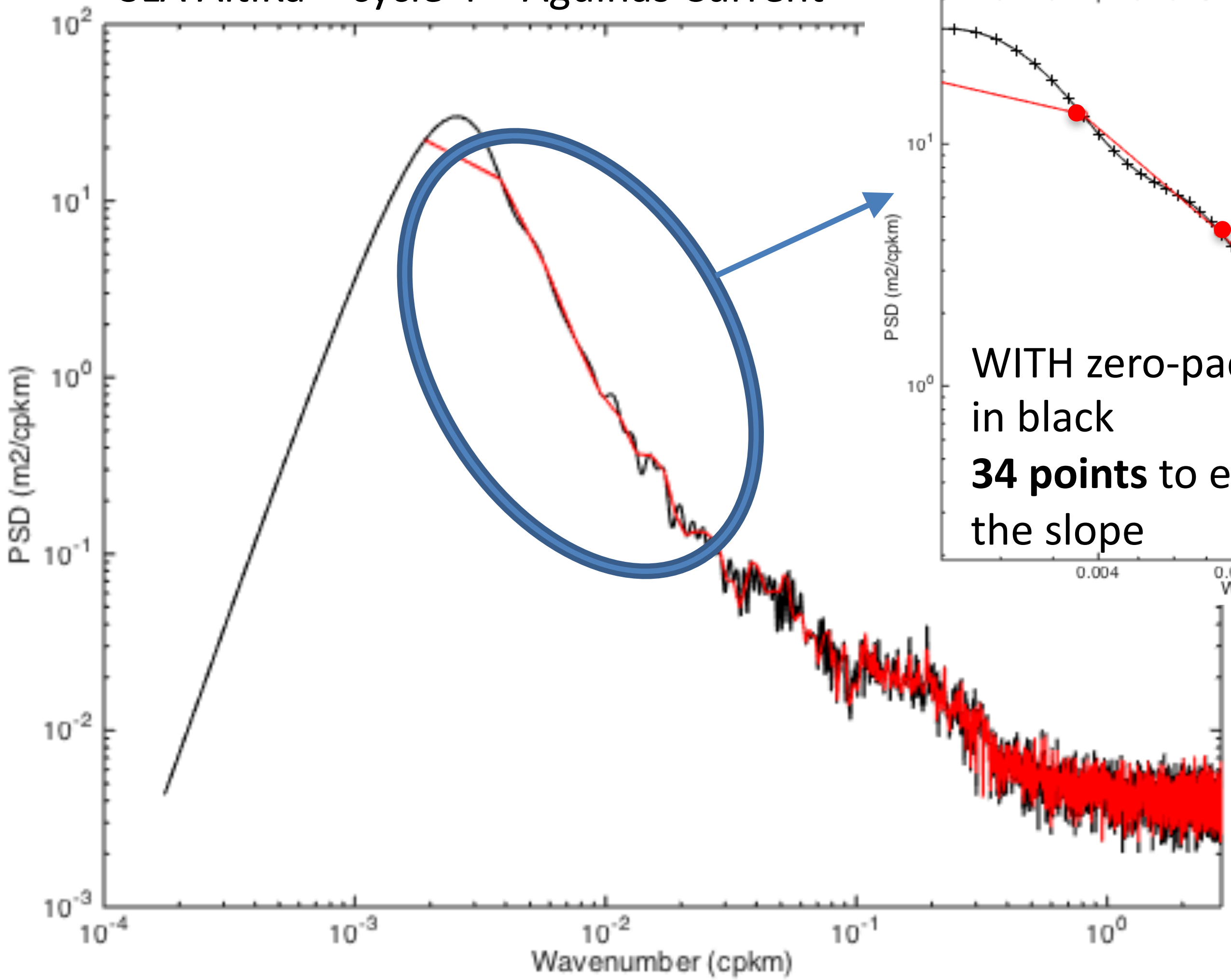
With ZERO PADDING



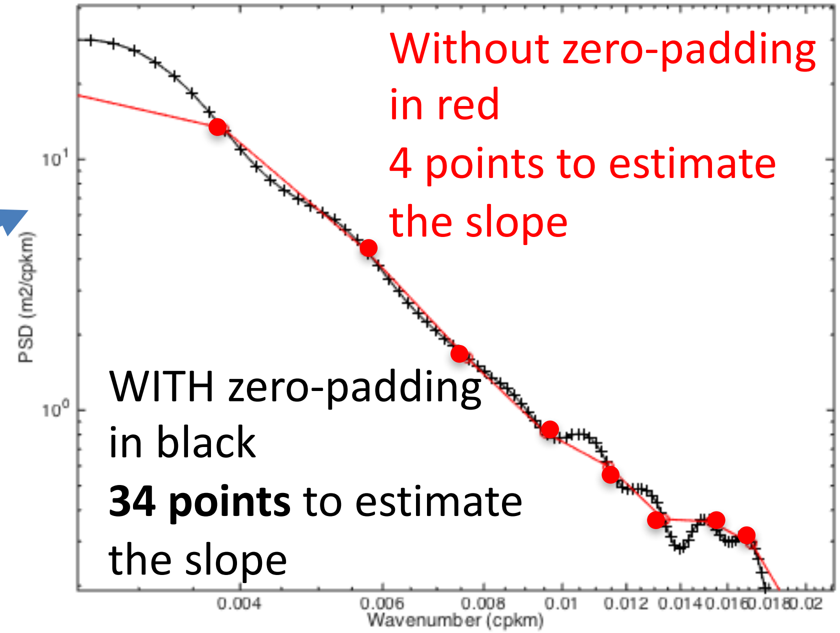
Supplementary slides

To illustrate what is ZERO-PADDING (interesting to better estimate the slope)

SLA AltiKa – cycle 4 – Agulhas Current



A better estimation of the slope



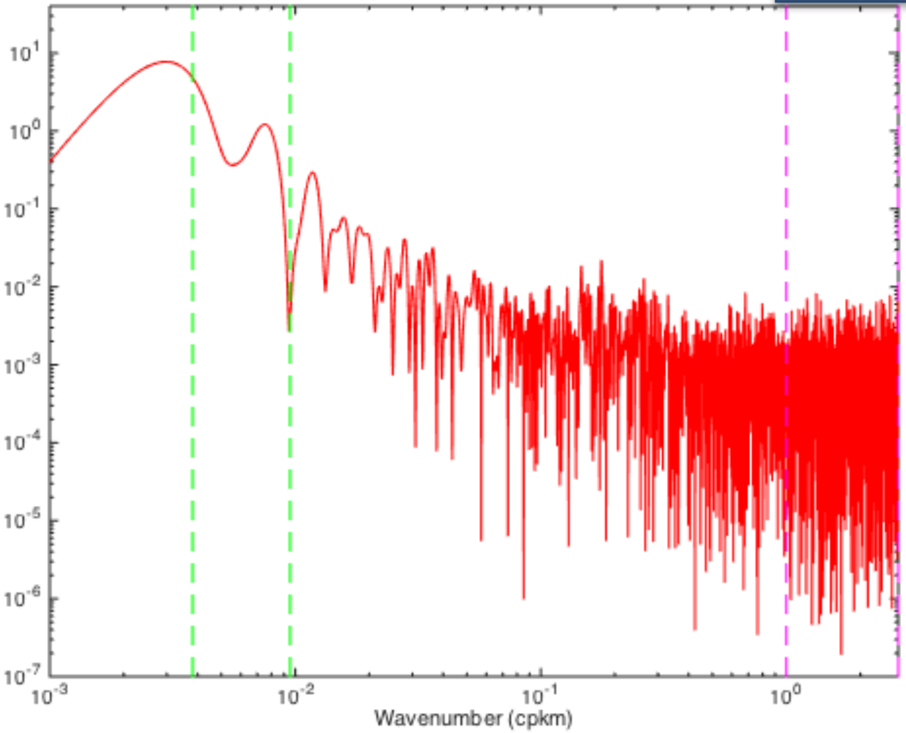
A better fit to estimate the slope

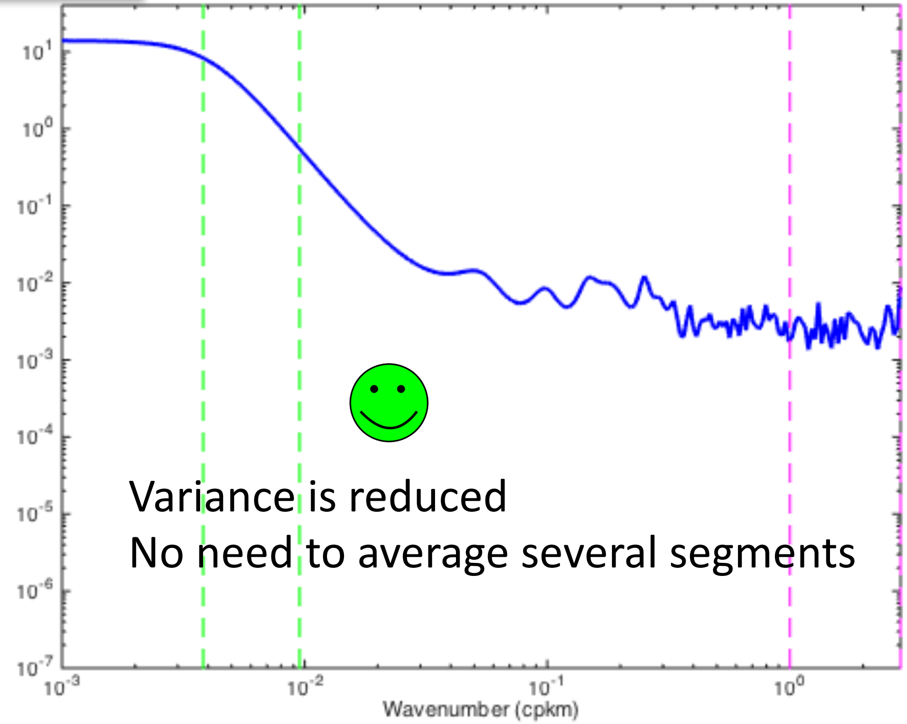
Supplementary slides

To illustrate
Fourier based spectral analysis v.s. AR spectral analysis

SLA AltiKa – cycle 4 –
Agulhas Current

1 segment
(3000 samples)





Periodogram
(1 segment of 3000 samples)
Zero-padding, detrend, rectangular window

AR spectral estimation (p=150)
(1 segment of 3000 samples)