Separating waves and eddies from sea surface height: theory, applications, and limitations

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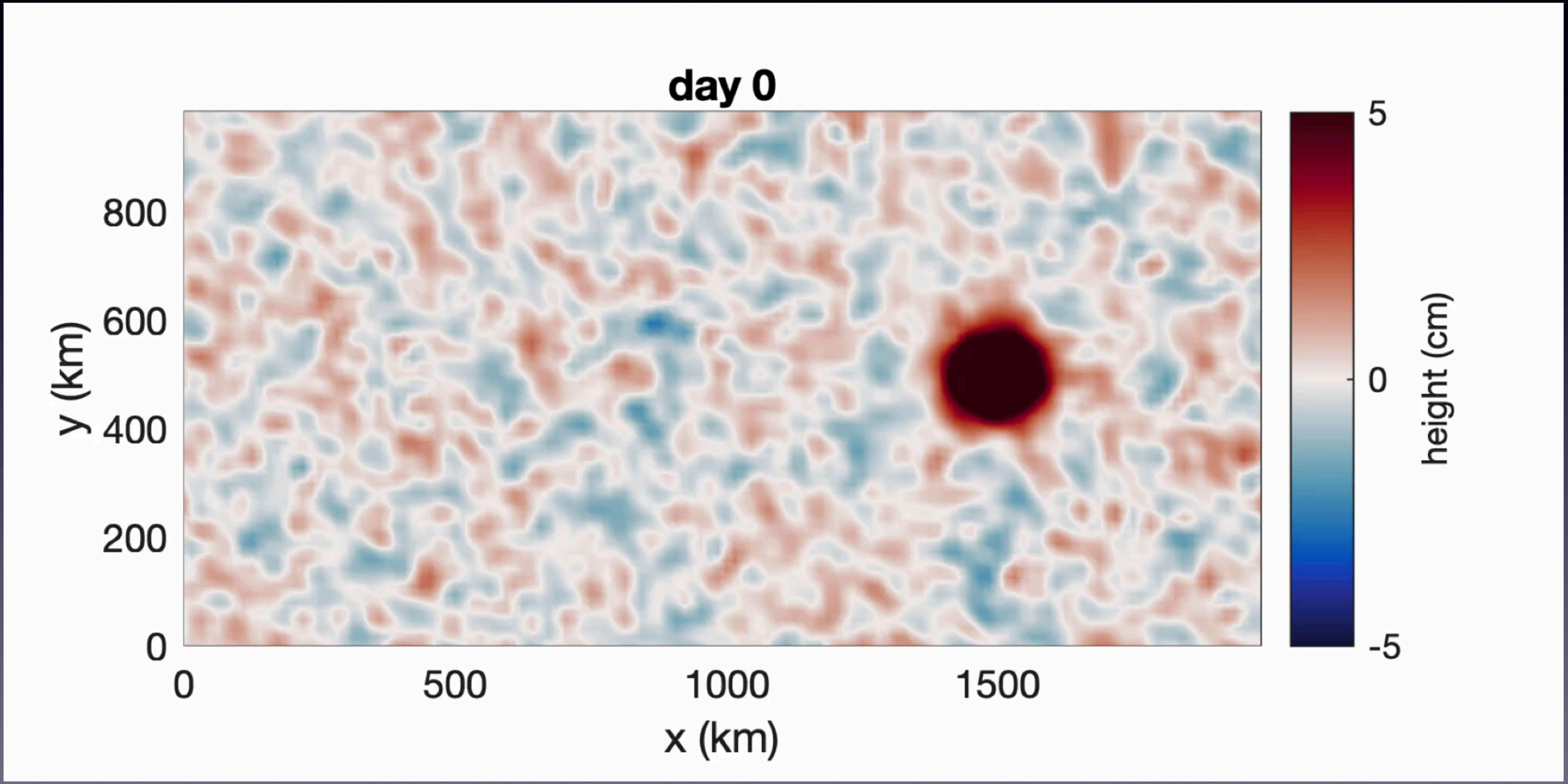
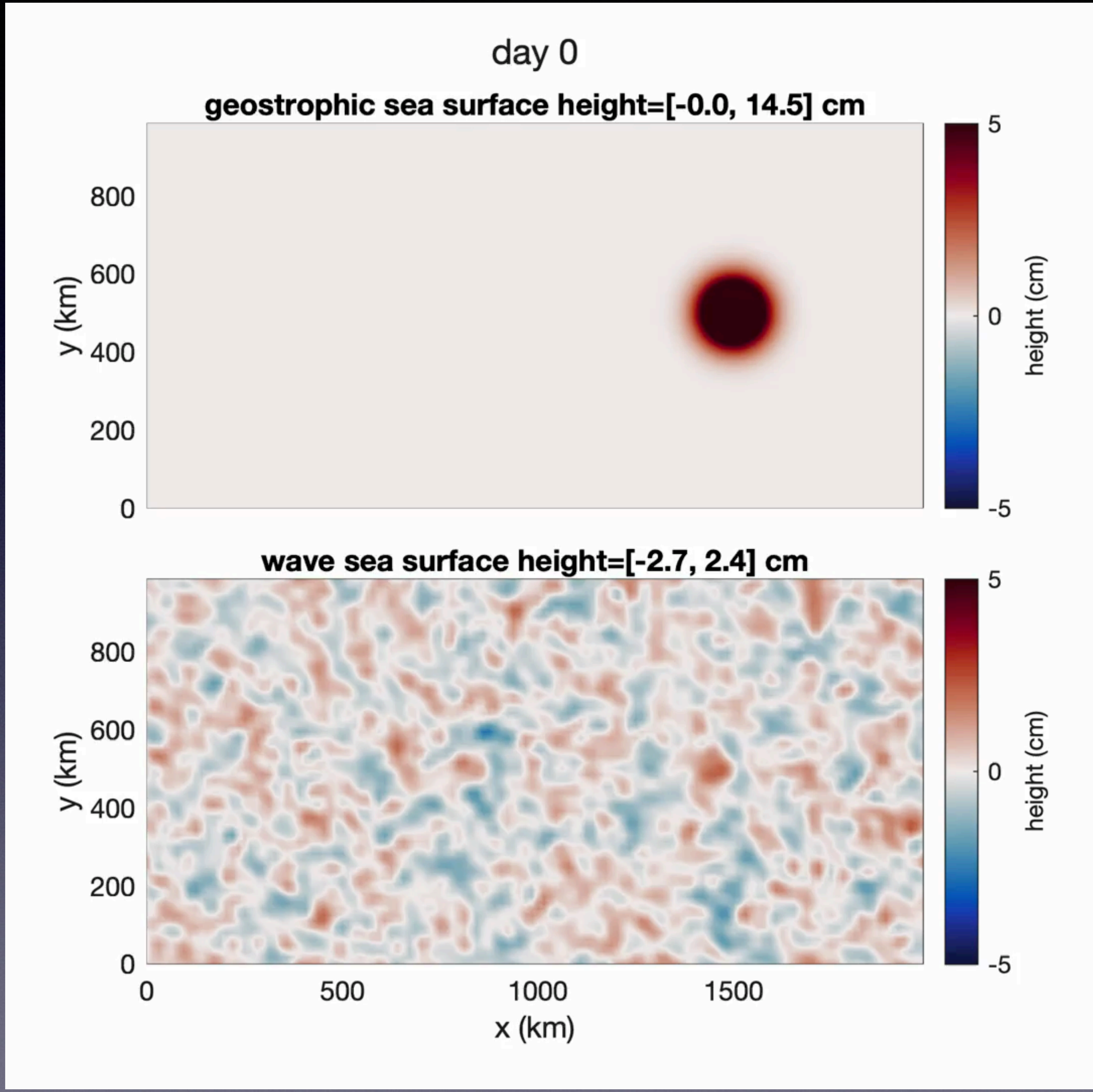
Outline

- The wave-vortex model
- The wave-vortex decomposition
- Dynamically parameter estimation from along track SSH
- Along track eddy analysis
- Limitations & forthcoming theory

SSH of IGW field + eddy



- The model decomposed *all* dynamical fields at each instant in time.
- Energy content of waves, inertial oscillations and geostrophic features always known.
- Absolutely no filtering in time.



Nonlinear equations of motion

$$
\begin{aligned}
\frac{d}{dt}u - f_0 v =& -\frac{1}{\rho_0}\partial_x p \\
\frac{d}{dt}v + f_0 u =& -\frac{1}{\rho_0}\partial_y p \\
\frac{d}{dt}w =& -\frac{1}{\rho_0}\partial_z p - \frac{1}{\rho_0} g\rho \\
\frac{d}{dt}\rho + w\partial_x \bar{\rho} =& 0 \\
\partial_x u + \partial_y v + \partial_z w =& 0
\end{aligned}
$$

- Non-hydrostatic, variable stratification
- Flat bottom, rigid lid

Early, Lelong, & Sundermeyer. *A generalized wave-vortex decomposition for rotating Boussinesq flows with arbitrary stratification.* ***JFM***, 2021.

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Wave-vortex solutions

Solutions

$$|\psi\rangle = \begin{bmatrix} u \\ v \\ \eta_e \end{bmatrix}$$

Geostrophic

$$|\Psi_g\rangle = \begin{bmatrix} -i\frac{g\ell}{f_0}F_g(z) \\ i\frac{gk}{f_0}F_g(z) \\ G_g(z) \end{bmatrix} e^{ikx+i\ell y}$$

Wave

$$|\Psi_{igw}\rangle = \frac{1}{\omega_w^j K} \begin{bmatrix} (k\omega_w^j \mp f_0 i\ell)F_w(z) \\ (\ell\omega_w^j \pm f_0 ik)F_w(z) \\ \mp K^2 h_w^j G_w(z) \end{bmatrix} e^{ikx+i\ell y \pm i\omega t}$$

$A_\pm$

j \ k	0	1	2	3
0	IO			
1	IO	IGW	IGW	IGW
2	IO	IGW	IGW	IGW
3	IO	IGW	IGW	IGW

A_0

j \ k	0	1	2	3
0		G^0	G^0	G^0
1	$\bar{\rho}'$	G	G	G
2	$\bar{\rho}'$	G	G	G
3	$\bar{\rho}'$	G	G	G

- IGW (internal gravity wave)
- G (internal mode geostrophic)
- IO (inertial oscillations)
- $\bar{\rho}'$ (mean density anomaly)
- G0 (barotropic geostrophic)

Wave Vortex Composition

- *All* physically realizable states of this system are a sum of these solutions.

$$|\psi\rangle = \sum_{klj} A_g^{klj} |\Psi_{\mathrm{g}}\rangle + A_w^{klj} |\Psi_{\mathrm{igw}}\rangle + \sum_{kl} A_{\mathrm{g0}}^{kl} |\Psi_{\mathrm{g0}}\rangle + \sum_{j} A_{\mathrm{io}}^{j} |\Psi_{\mathrm{io}}\rangle + A_{\mathrm{mda}}^{j} |\Psi_{\mathrm{mda}}\rangle + \mathrm{c.c.}$$

Early, Lelong, & Sundermeyer. *A generalized wave-vortex decomposition for rotating Boussinesq flows with arbitrary stratification.* ***JFM***, 2021.

Energy Orthogonality

To diagnose whether energy went ***from*** somewhere ***to*** somewhere else your decomposition must be energetically orthogonal.

$$\frac{1}{2L_xL_y}\int_{-D}^{0}\int_{0}^{L_y}\int_{0}^{L_x}(u^2+v^2+w^2+N^2\eta^2)\,dx\,dy\,dz = \sum_{jkl}\alpha_{jkl}A_+^2+\alpha_{jkl}A_-^2+\beta_{jkl}A_0^2$$

Parseval's theorem, for wave-vortex energy and enstrophy

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Wave Vortex Model

- Initialize a model with variable stratification

```
N2 = @(z) N0*N0*exp(2*z/L_gm);
wvt = WVTransformHydrostatic([Lx, Ly, Lz], [Nx, Ny, Nz], N2=N2,latitude=25);
```

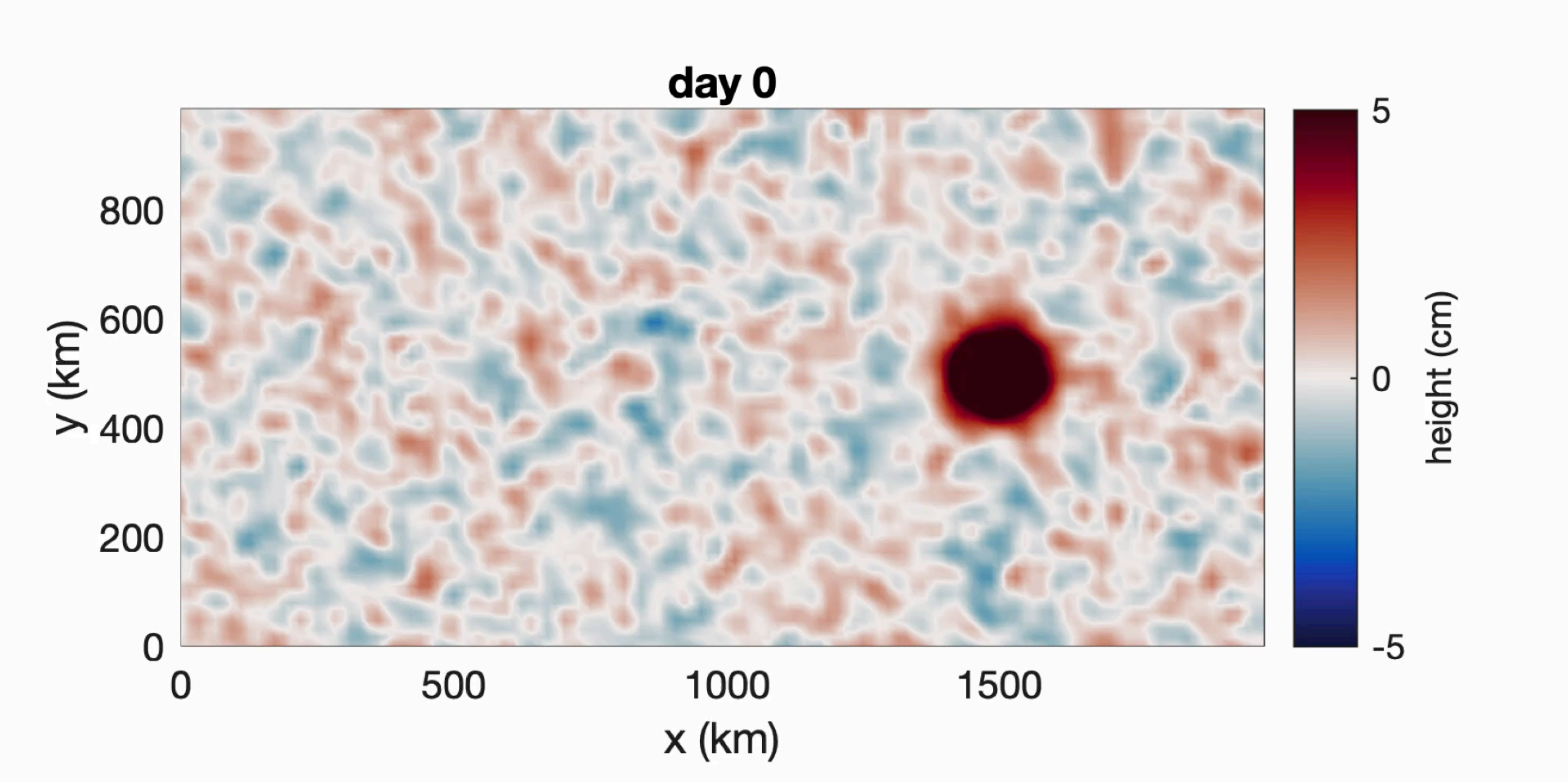
- Add an internal wave field and an eddy

```
wvt.initWithGMSpectrum(1.0);
psi = @(x,y,z) U*(Le/sqrt(2))*exp(1/2)*exp(-((x-x0)/Le).^2 -((y-y0)/Le).^2 -(z/He).^2 );
wvt.setGeostrophicStreamfunction(psi);
```

- Run the model for a year

```
model = WVModel(wvt,nonlinearFlux=WVNonlinearFlux(wvt,shouldUseBeta=1,uv_damp=wvt.uMax);
model.integrateToTime(365*86400);
```

SSH of IGW field + eddy



Wave Vortex Model

- Equations of motion time-stepped in wave-vortex space.
- Each degree-of-freedom in the model has a dynamical interpretation.
- Nonlinear terms ‘reshuffle’ energy.
- Transfer mechanism always have a physical interpretation.
- Computational efficient for variable stratification.

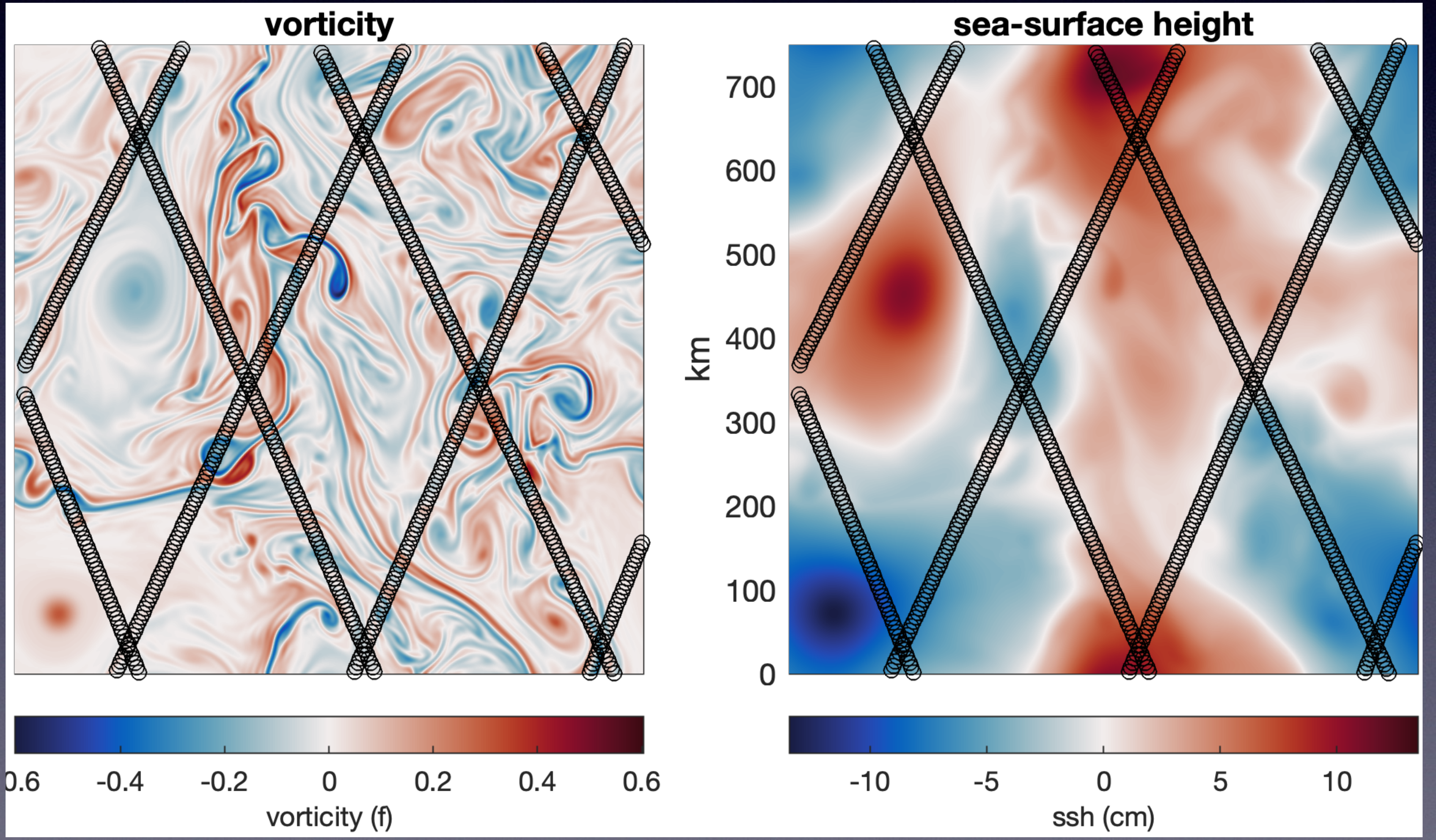
Sea-surface height mapping

Cim Wortham & Arthur Guillaumin

- dynamically aware statistical model for sea-surface height

    1. Improve gridded interpolation of ssh

    2. Estimate physically relevant parameters

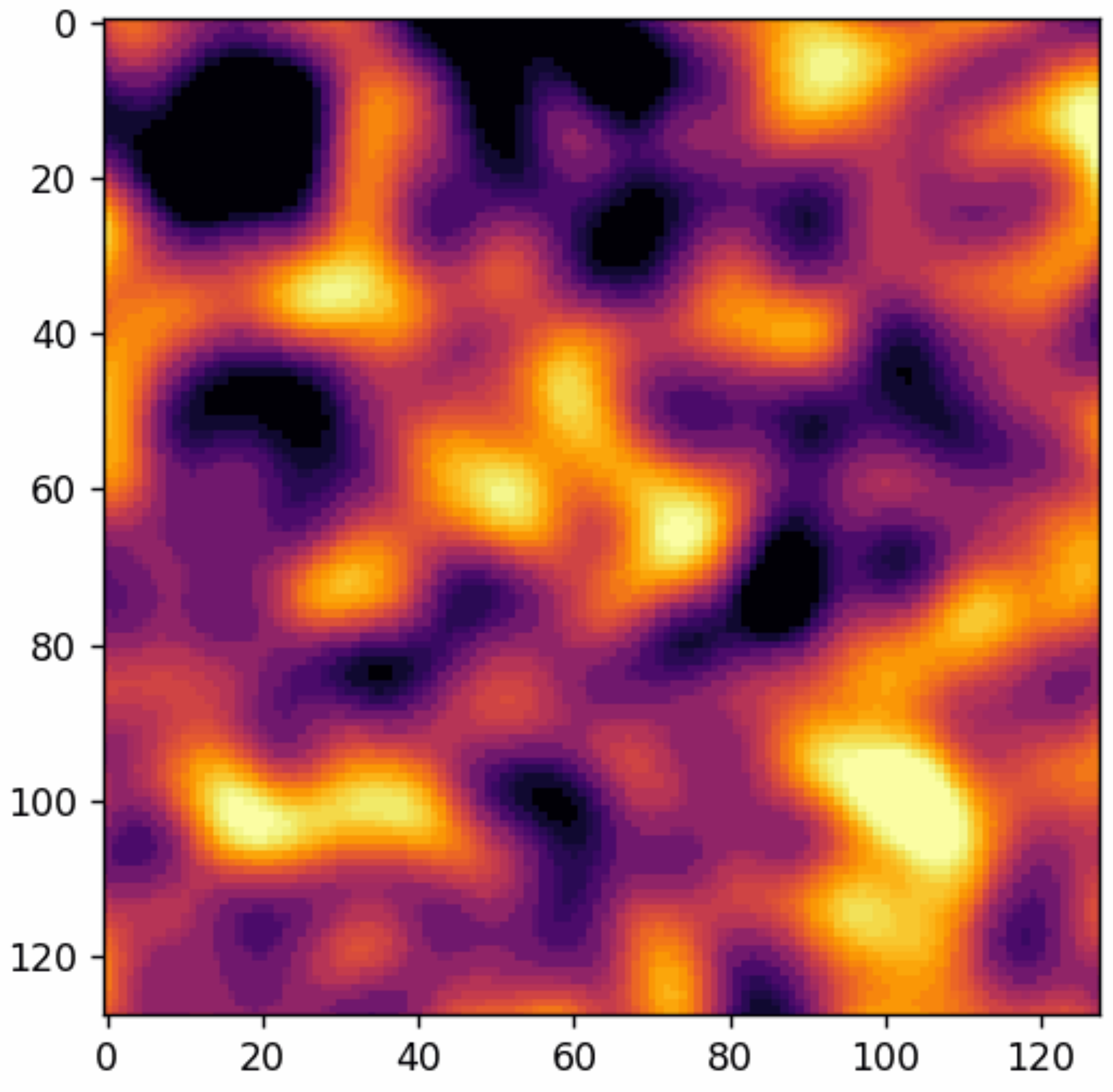


Turbulence parameters: β_0, k_f, r, L_r

Stochastic parameters: σ, L_x, L_y, c_y, T

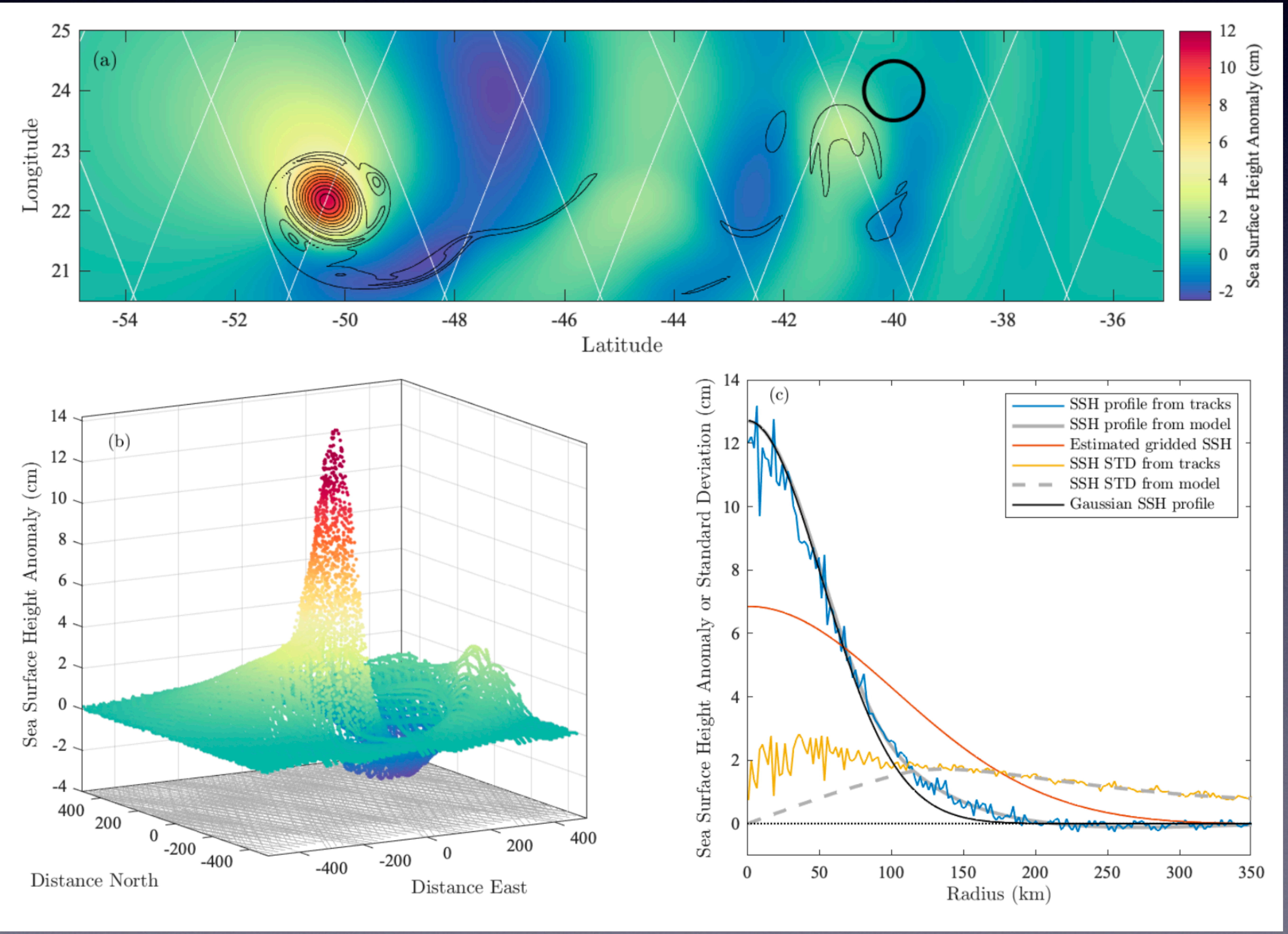
- Stochastic model has parameter which can (sometimes!) be related to the physical parameters
- Example by Arthur Guillaumin at Queen Mary, University of London

$$\mathrm{cov}(x,y,t) = \sigma^2 \exp\left(-\frac{x^2}{L_x^2} - \frac{(y-c_y)^2}{L_y^2} - \frac{t^2}{T^2}\right)$$



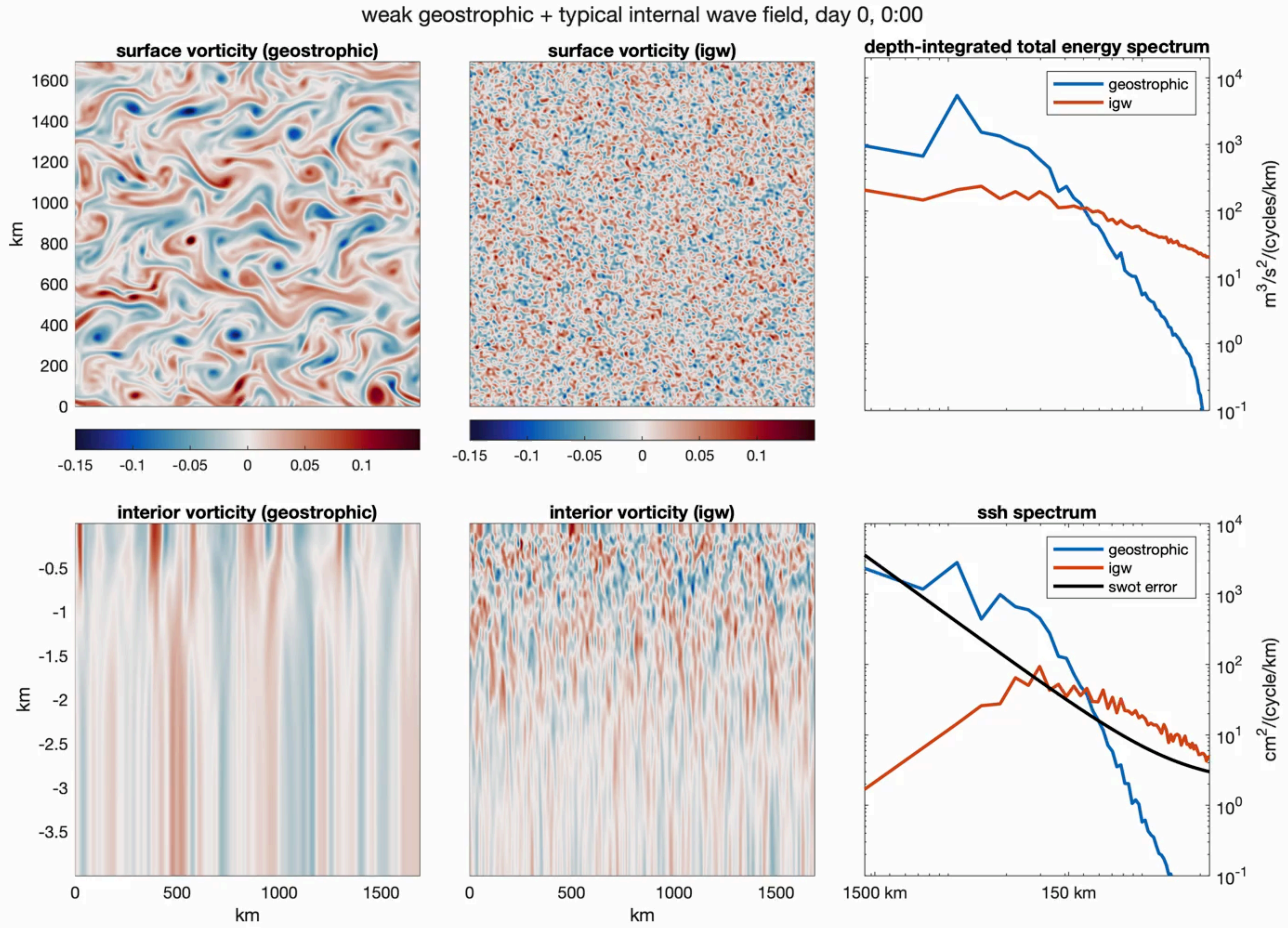
Along track eddy detection

Jonathan Lilly, Pete Gaube, & Chris Ohh



Great tool for altimetry

- The sea-surface height is instantaneously decomposed into waves and geostrophic motions.
- Setup realistic box simulations
  - Realistic stratification—details affect ssh expression (Gaube)
  - Baroclinic instability forces mesoscale field
  - Wind and tides forces IGW field
- Dynamics are known (e.g., energy and enstrophy fluxes)

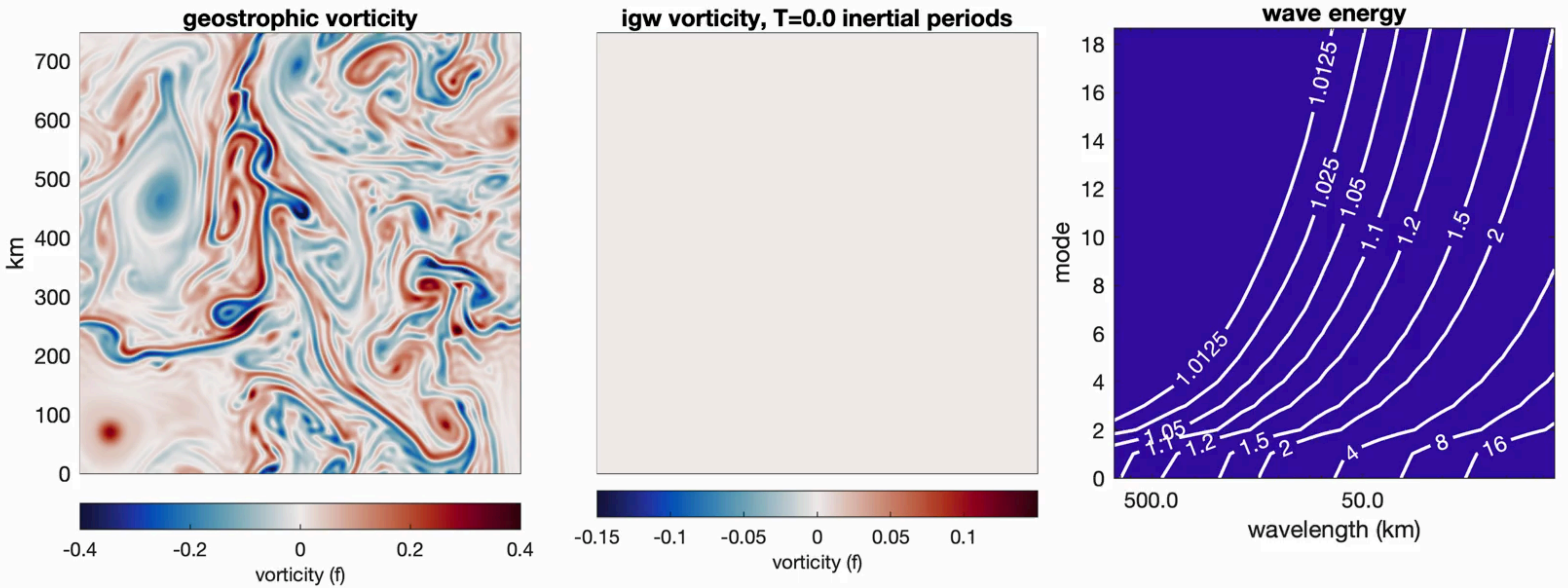

weak geostrophic + typical internal wave field, day 0, 0:00
surface vorticity (geostrophic)
surface vorticity (igw)
depth-integrated total energy spectrum
geostrophic
igw
$m^3/s^2/(cycles/km)$
interior vorticity (geostrophic)
interior vorticity (igw)
ssh spectrum
geostrophic
igw
swot error
$cm^2/(cycle/km)$
km
1500 km
150 km

Looking forward

- Decomposition hinges on QGPV—which is not a good approximation to PV in many regimes. Available Potential Vorticity tells you when this occurs, Early, et al. (2022)

- Methodology now extended to include surface buoyancy anomalies and an explicit free-surface. Work with Gerardo Hernández-Dueñas, Leslie Smith and Pascale Lelong.

- Decomposition provides a method for inferring interior flow from SSH and SST, following the basic idea of Wang et al. (2013)



geostrophic vorticity
km
0
100
200
300
400
500
600
700
-0.4
-0.2
0
0.2
0.4
vorticity (f)
igw vorticity, T=0.0 inertial periods
-0.15
-0.1
-0.05
0
0.05
0.1
vorticity (f)
wave energy
mode
0
2
4
6
8
10
12
14
16
18
500.0
50.0
wavelength (km)
1.0125
1.025
1.05
1.1
1.2
1.5
2
4
8
16