Optimal Parameters for Mapping Along-Track Altimetry

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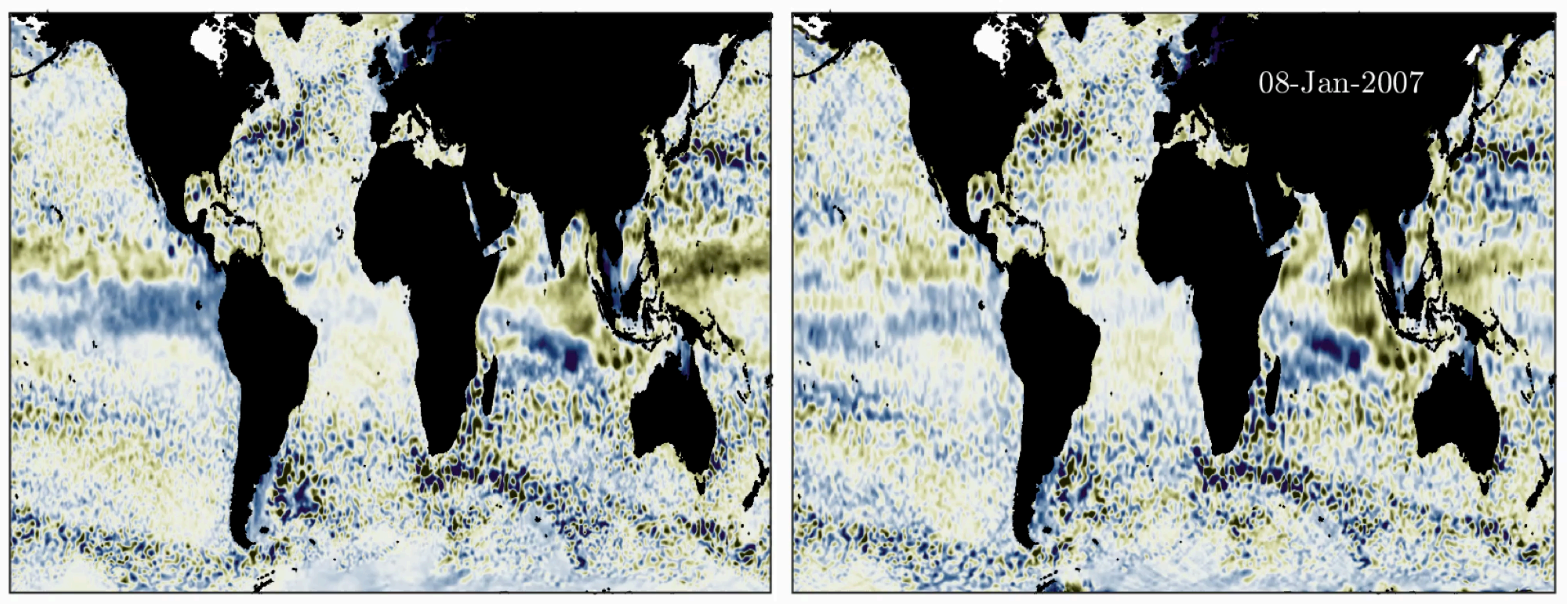
An Open-Source Gridded Product



From Lilly (2023), in prep.

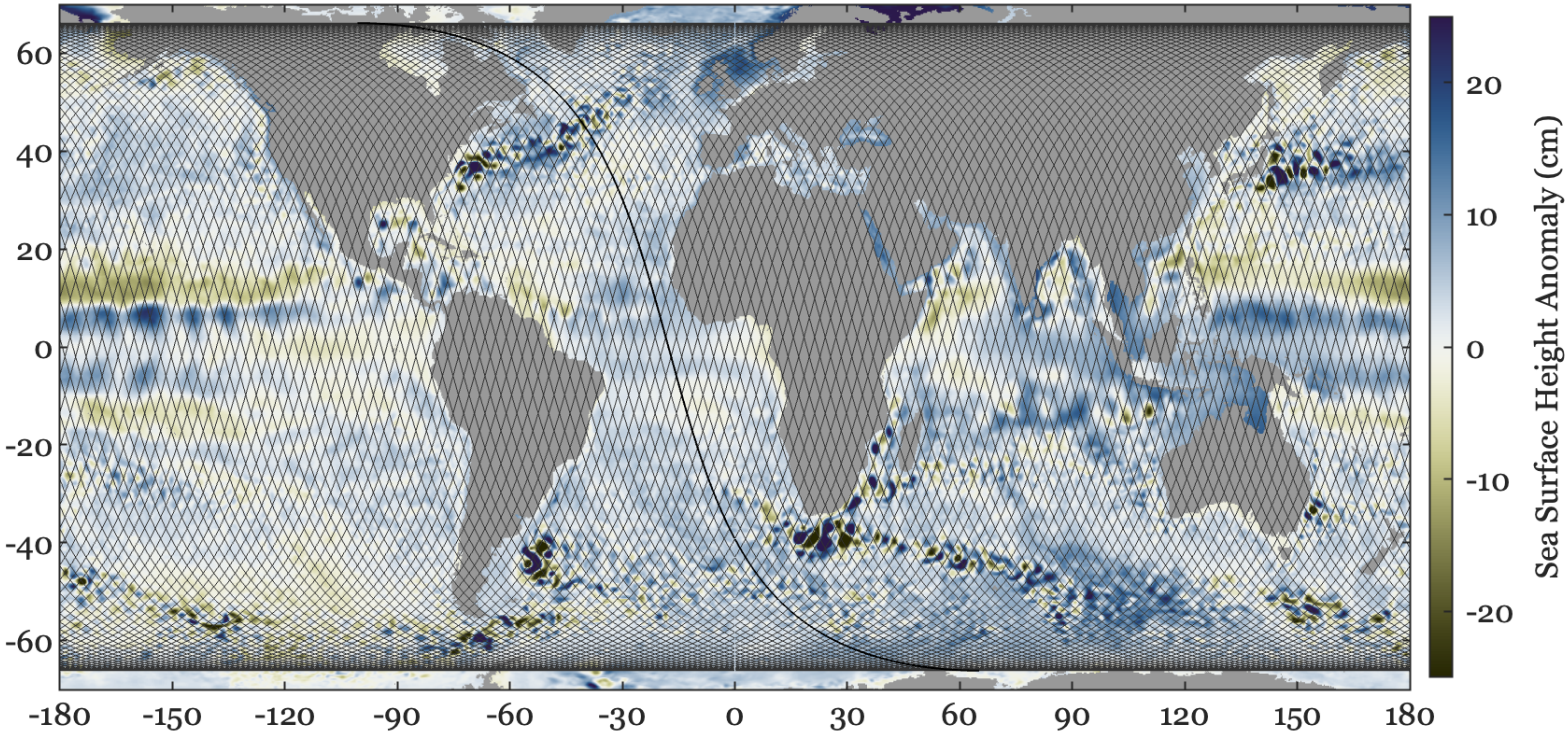
Determining Optimal Mapping Parameters

We will determine optimal parameters for mapping Jason-class satellite altimetry. This is essentially a study in mapping theory applied to a particular problem.

The main framework is an OSSE in which a general circulation model (GOLD-NoTides, Simmons and Alford, 2012) is observed with realistic altimeter-like sampling.

- Jason-class altimetry only, for simplicity
- Realistic small-scale noise, inferred from along-track data, is added to the model.
- Assumption of synopticity (the 9.92-day cycle mean field is observed instantaneously)
- Maps created using local polynomial fitting
- Isotropic (i.e. circular, non-elliptical) weighting kernel for now
- Best choice of parameters is founding by minimizing MSE vs. model truth in a large parameter sweep (>40,000 maps) followed by optimizations.

A Synthetic Along-Track Dataset



From a GOLD-derived data product by Simmons and Lilly (2023), in prep.

What is Local Polynomial Fitting?

This is simply a least squares fit, at each mapping point, of a polynomial to the data, with a weighting function concentrating the effect of nearby data points.

- The fit *order, P,* determines whether we fit to a constant ($P = 0$), a plane ($P = 1$), or a parabolic surface ($P = 2$). Note the $P = 0$ case is the same as a kernel smoothing.
- The weighting function, denoted $K(r)$, is called the *kernel*. $K(r)$ vanishes for $r > 1$.
- The kernel is spatially rescaled by the *bandwidth h* to give $K_H(r) \equiv H^{-1}K(r/H)$.
- The number of data points within the nonzero portion we will call the *population N*.
- Two different variants are *fixed bandwidth* (B=const) or *fixed population* (N=const).
- The distribution of data points is called the *design;* as in, the design of an experiment.

What is Local Polynomial Fitting?

Let (x_o, y_o) be a mapping point and $\widetilde{\mathbf{z}}$ be a array of data collected at locations $\widetilde{\mathbf{x}}, \widetilde{\mathbf{y}}$.

We seek to minimize the weighted least squares error

$$\Delta_P^2(x_o, y_o) \equiv \left\| \mathbf{W}\left(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}, x_o, y_o\right) \left[\tilde{\mathbf{z}} - \mathbf{X}_P\left(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}, x_o, y_o\right) \widehat{\mathbf{z}}_P\left(x_o, y_o\right)\right] \right\|^2$$

where the array $\widehat{\mathbf{z}}_P\left(x_o, y_o\right)$ is the estimated field value and its first P derivatives, $\mathbf{X}_P$ is matrix of up to Pth order polynomials in $\tilde{\mathbf{x}} - x_o$ and $\tilde{\mathbf{y}} - y_o$, and $\mathbf{W}$ is a weighting matrix

$$\mathbf{W}\left(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}, x_o, y_o\right) \equiv \operatorname{diag}\left\{\begin{bmatrix} K_H\left(\tilde{r}_1\right) & K_H\left(\tilde{r}_2\right) & \ldots & K_H\left(\tilde{r}_N\right)\end{bmatrix}\right\}.$$

The solution is given by

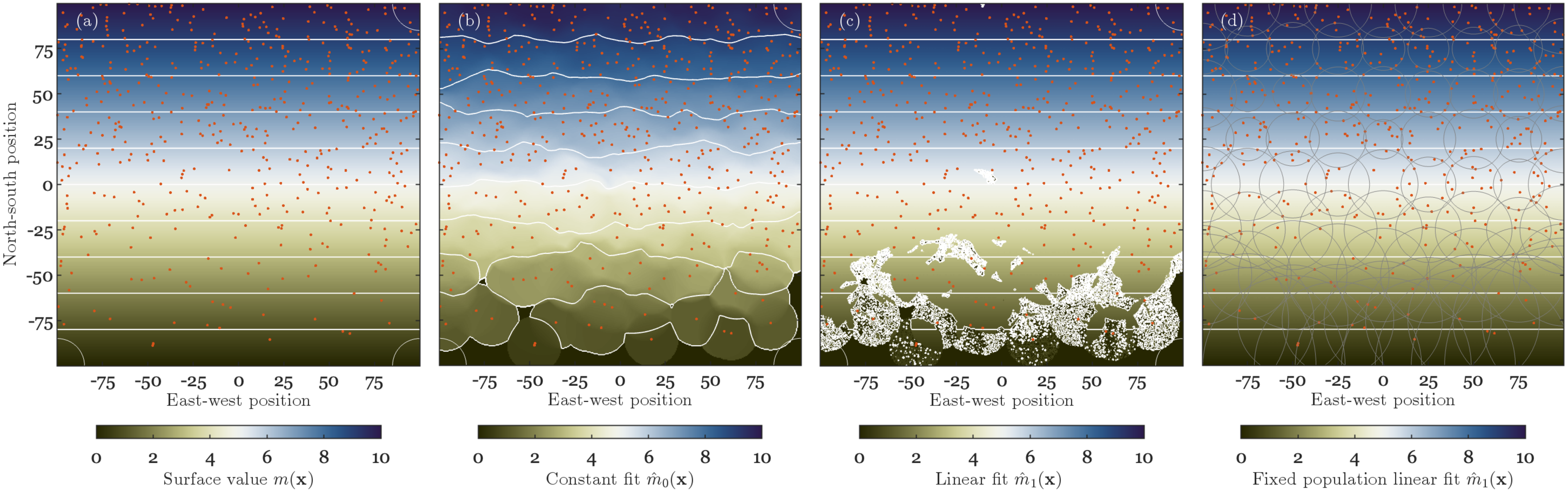
$$\widehat{\mathbf{z}}_P\left(x_o, y_o\right) = \left(\mathbf{X}_P^T \mathbf{W} \mathbf{X}_P\right)^{-1} \mathbf{X}_P^T \mathbf{W} \tilde{\mathbf{z}}$$

where the matrix to be inverted is 1×1 for $P = 0$, 3×3 for $P = 1$, and 6×6 for $P = 2$.

Advantages of Local Polynomial Fitting

- Simple: just choose an order, weighting kernel, and bandwidth settings
- Easy to see and describe what is being done with the data
- Huge body of theoretical results for parameter choosing & performance evaluation
- Parsimonious: no need to assume the covariance properties of the mapped field
- Velocity (for $P \geq 1$) and vorticity (for $P \geq 2$) are intrinsically also estimated
- Natural extendable to elliptical smoothing regions
- Easily augmented by a robustification step (Cleveland, 1979)
- Fast: a rate-limiting matrix inversion is 1-2 orders of magnitude smaller than in OI
- Can increase resolution as data density increases, unlike OI
- As a linear method, should be able to match the performance of any linear method

Design Adaptivity



Mapping a gradient from non uniformly distributed data samples. Two key take-aways.

(i) First and higher-order fits are *design-adaptive* (Fan, 1992) — there is no bias due to nonuniform design. Note, one should never use $P = 0$!

(ii) Fixed-population (a.k.a. variable bandwidth) + $P \geq 1$ fits are even more adaptive.

Choices of Weighting Kernel

1. Uniform:

$$K(r) = 1$$

2. Parabolic / Epanechnikov (Epanechnikov, 1969):

$$K(r) = 1 - r^2$$

3. Bisquare (Brundsdon et. al, 1996):

$$K(r) = (1 - r^2)^2$$

4. Tricube (Cleveland and Devlin, 1988):

$$K(r) = (1 - r^3)^3$$

5. Truncated Gaussian (Schlax et al., 2001):

$$K(r) = e^{-\frac{1}{2}\frac{r^2}{L^2}}$$

All are defined to vanish for $r > 1$, and ignoring a normalizing constant.

A Unified Family of Weighting Kernels

Proposing the *beta kernel* as a unified family of weighting functions,

$$K_{\beta,\gamma}(r) \equiv (1 - r^{\gamma})^{\beta}$$

controlled by the two parameters β and γ.

Explicitly includes 1–4 from previous slide as special cases, and closely approximates 5.

Includes all commonly-used forms, allowing for (i) exact computations of theoretical properties and (ii) parameter sweeps through a broad class of weighting kernels.

See Lilly (2023), in prep.

Moment of the Beta Kernel

Integral moments needed to deduce the asymptotic behavior of maps generated with the beta kernel can be readily solved for.

$$\mu_2 \equiv \pi \int_0^1 r^3 K(r) \mathrm{d}r = \frac{1}{2} \frac{B\left(4/\gamma, \beta + 1\right)}{B\left(2/\gamma, \beta + 1\right)}$$

$$\nu_0 \equiv 2\pi \int_0^1 r K^2(r) \mathrm{d}r = \frac{\gamma}{2\pi} \frac{B\left(2/\gamma, 2\beta + 1\right)}{B^2\left(2/\gamma, \beta + 1\right)}$$

Asymptotic error $\mathscr{E} \sim \left(\mu_2 \nu_0\right)^{2/3}$

(Ruppert and Wand, 1994; Fan and Gijbels 1996)

“Asymptotic” in this case means in the limit of a large number of observational points.

The Inner Radius

The beta kernel can be reparameterized in terms of the *inner radius* $R_{\beta,\gamma}$

$$K_{\beta,\gamma}(r) \equiv (1 - r^{\gamma})^{\beta} = \frac{1}{2}, \qquad\qquad r = R_{\beta,\gamma} = \left(1 - \frac{1}{2^{1/\beta}}\right)^{1/\gamma}$$

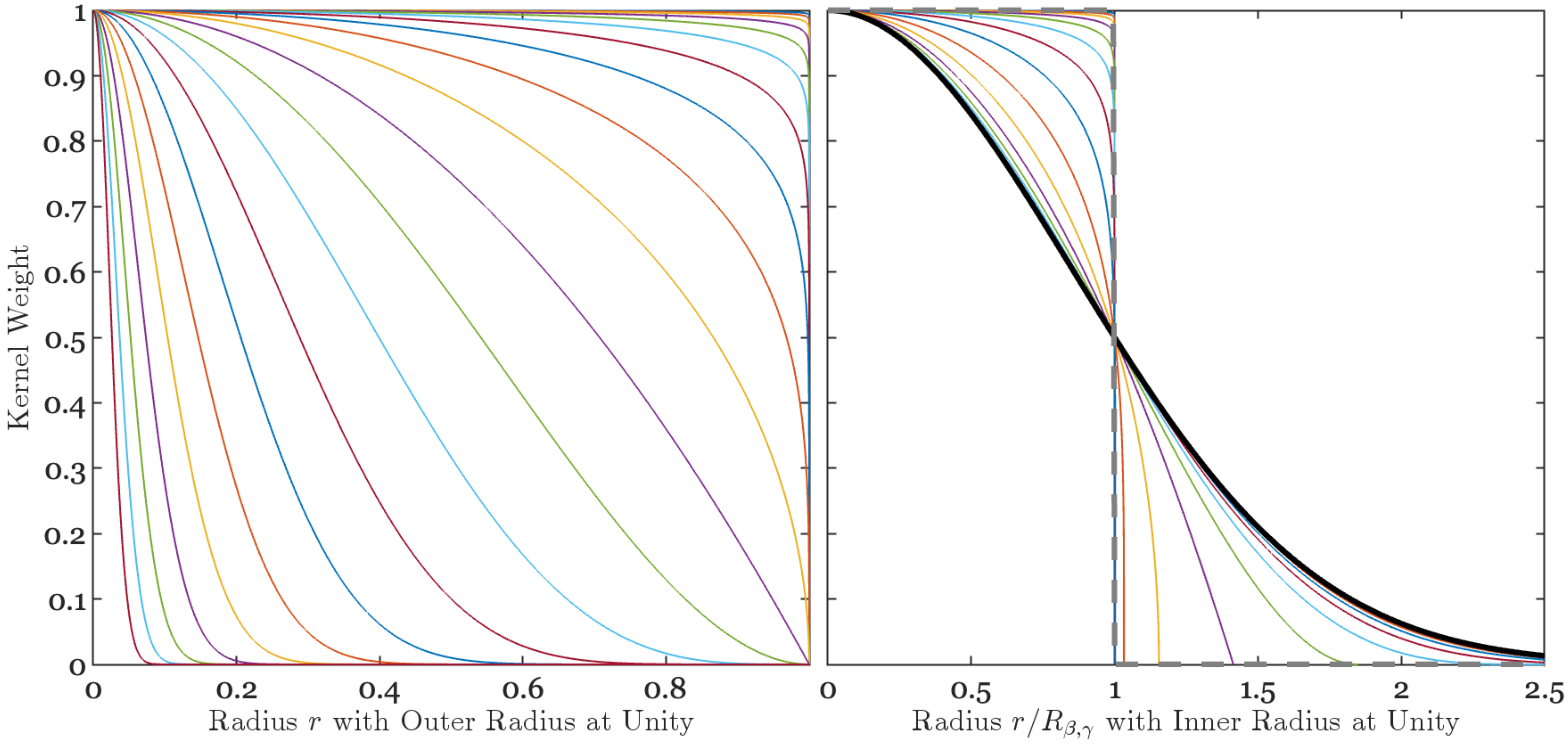
defined as the kernel's half-power point. Considering γ and $R_{\beta,\gamma}$ as fixed, we obtain β as

$$\beta = \frac{\ln \frac{1}{2}}{\ln\left(1 - R_{\beta,\gamma}^{\gamma}\right)}.$$

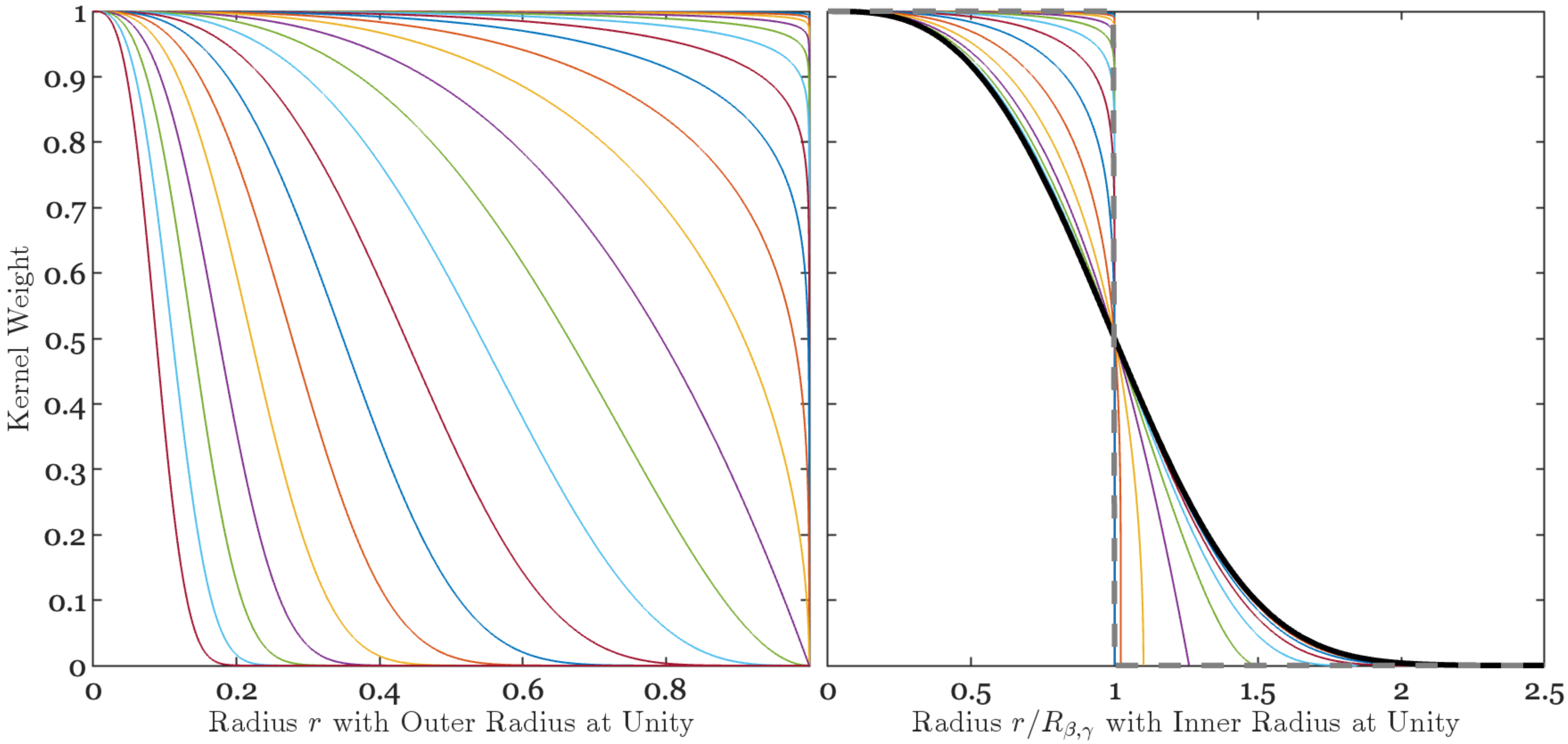
We can then consider the beta kernel to be a function of the inner radius together with γ, which we can term the *shape parameter*. For large $R_{\beta,\gamma}$ or β, the beta kernel behaves as

$$\lim_{\beta \longrightarrow \infty} K_{\beta,\gamma}(r) \sim \mathrm{e}^{-\frac{1}{2}\left|\left(2\log 2\right)^{1/\gamma}\frac{r}{R_{\beta,\gamma}}\right|^{\gamma}}.$$

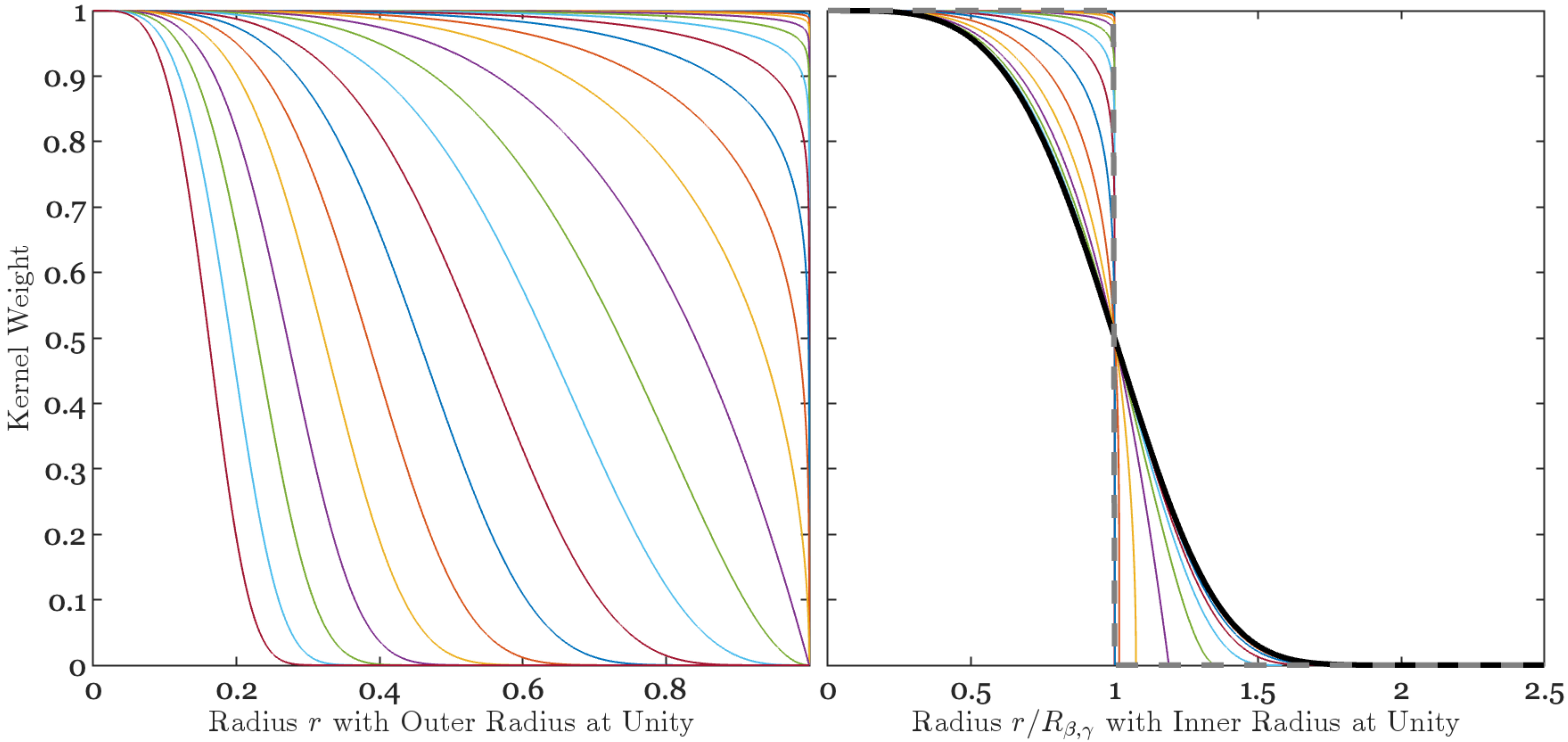
The $\gamma = 2$ Family with Varying β



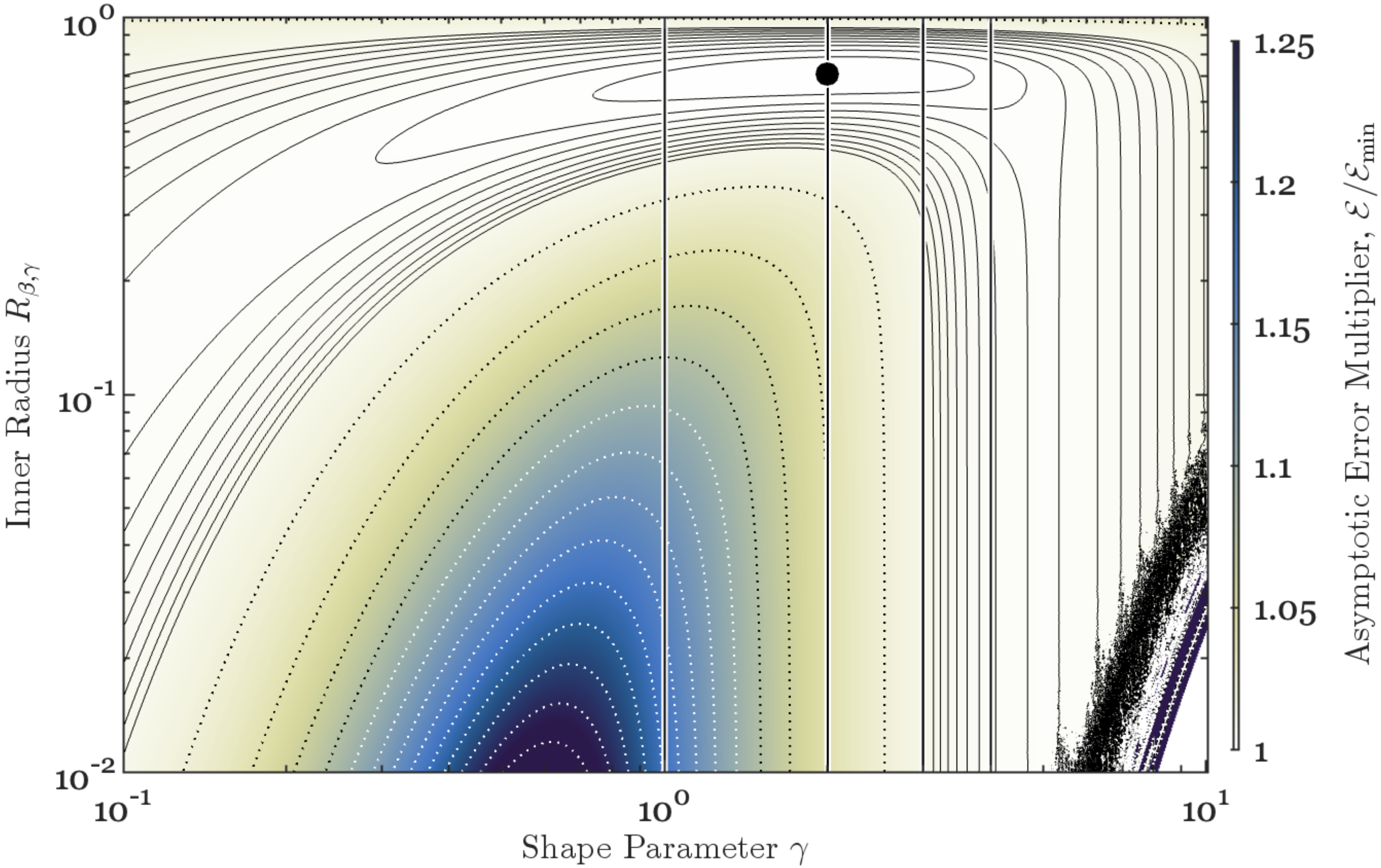
The $\gamma = 3$ Family with Varying β



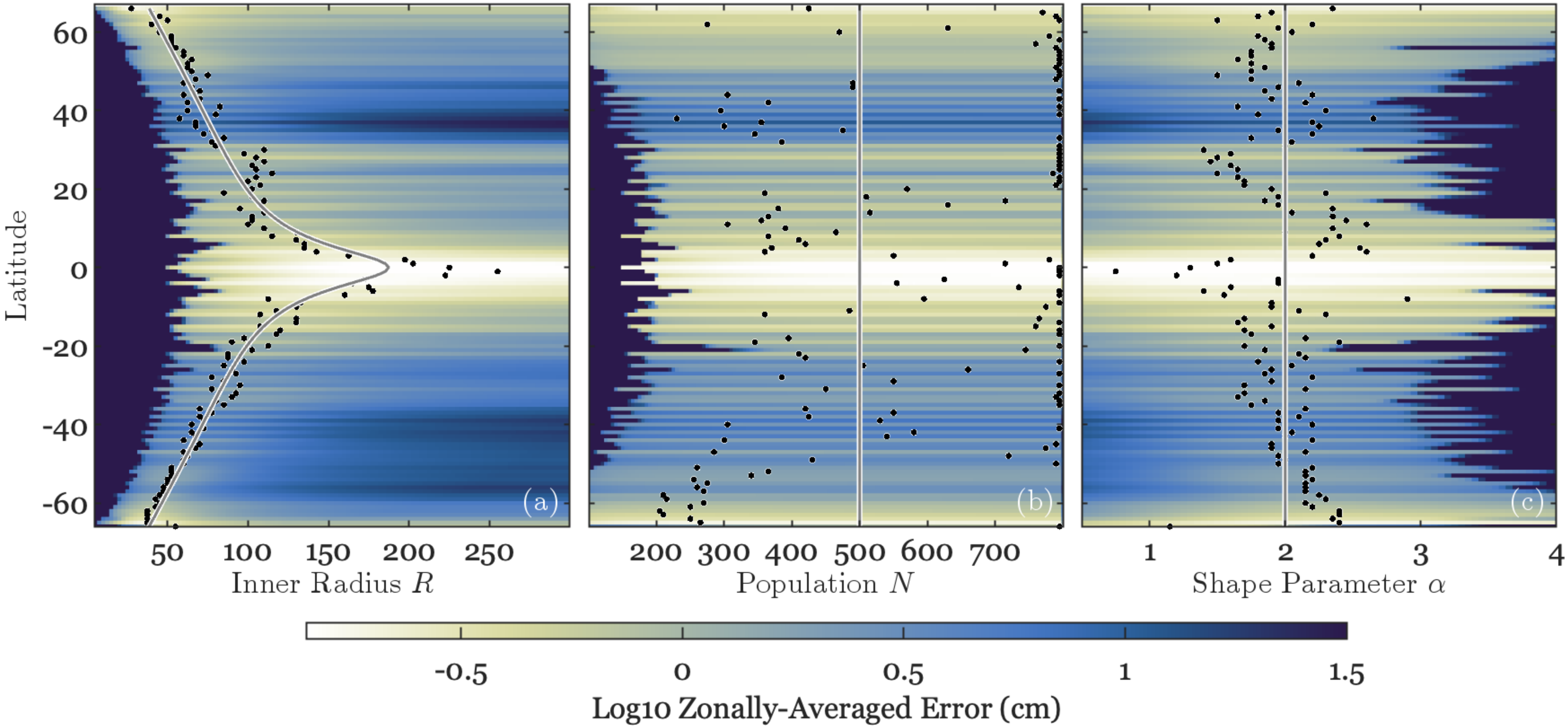
The $\gamma = 4$ Family with Varying β



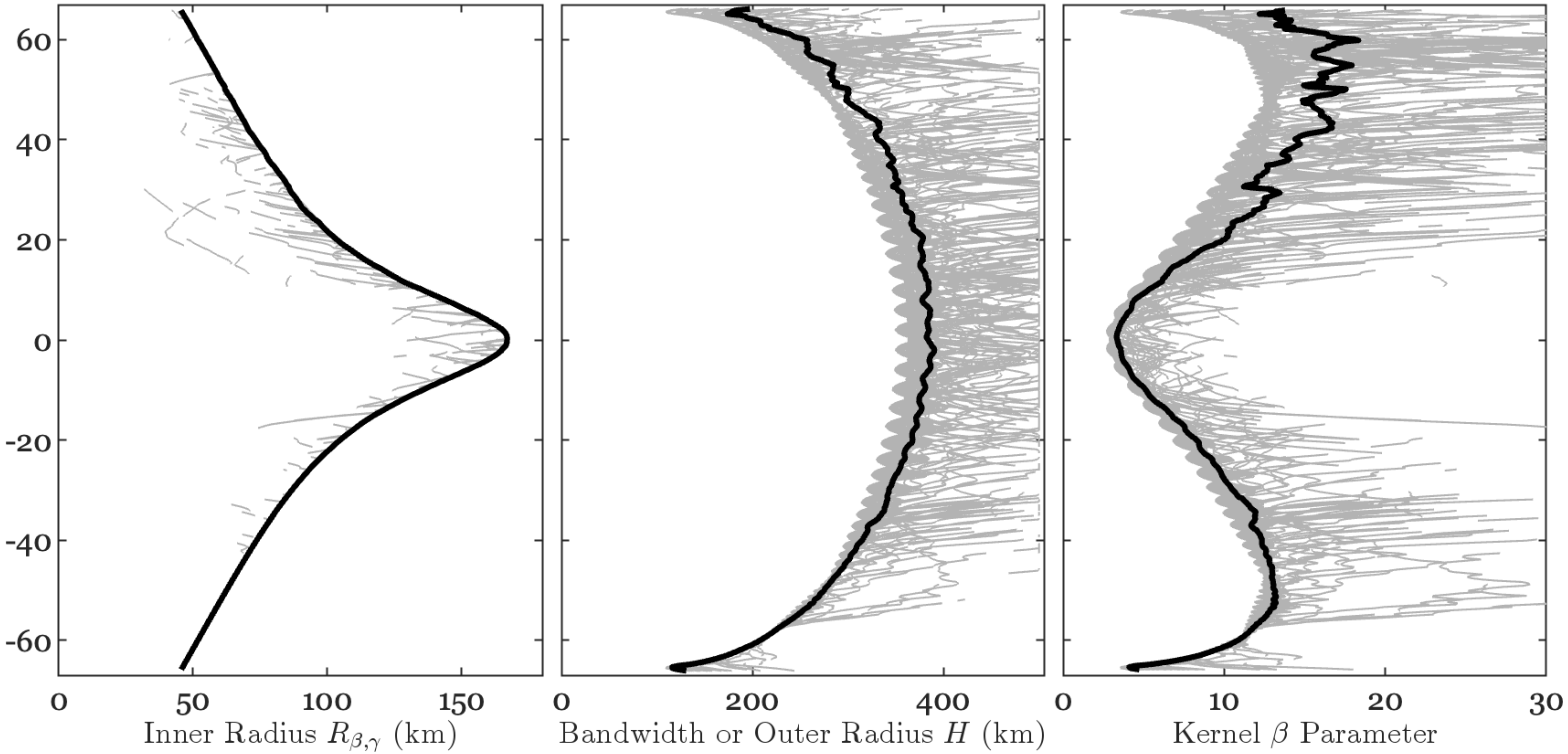
Asymptotic Error of Beta Kernel



Optimal Latitude-Dependent Parameters

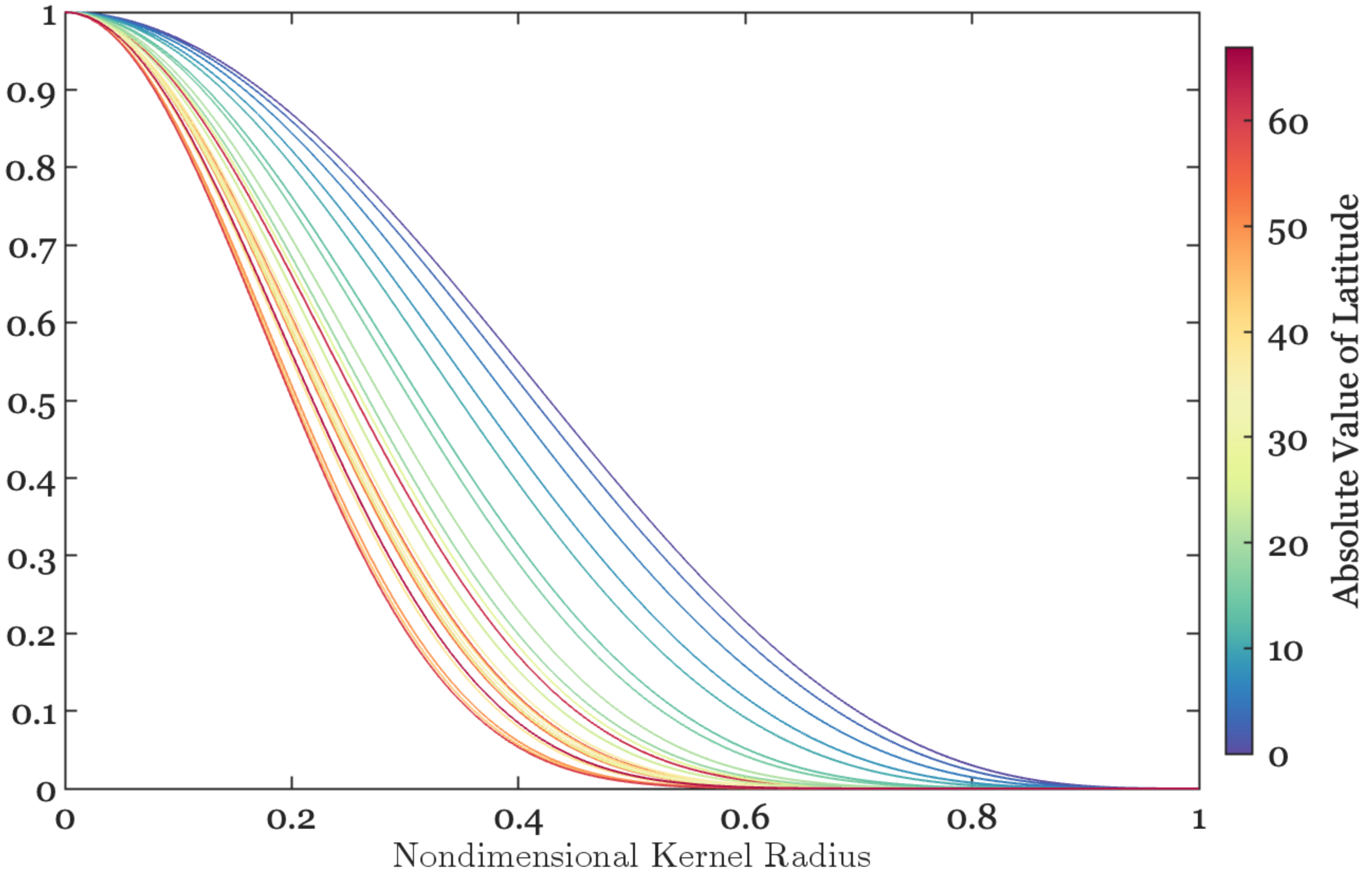


Latitudinally Varying Inner Radius

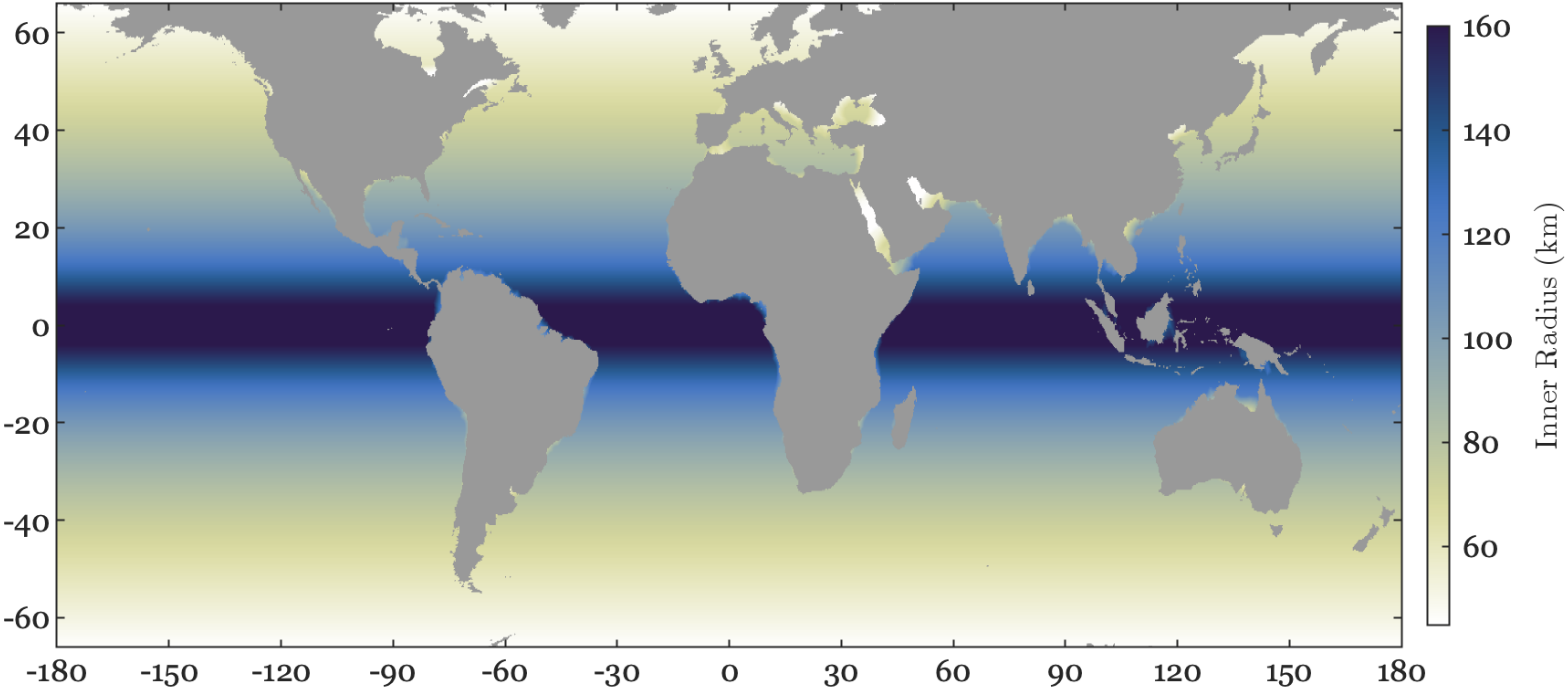


Fixed population $N = 500$, fixed $\gamma = 2$, geographically varying bandwidth H and β.

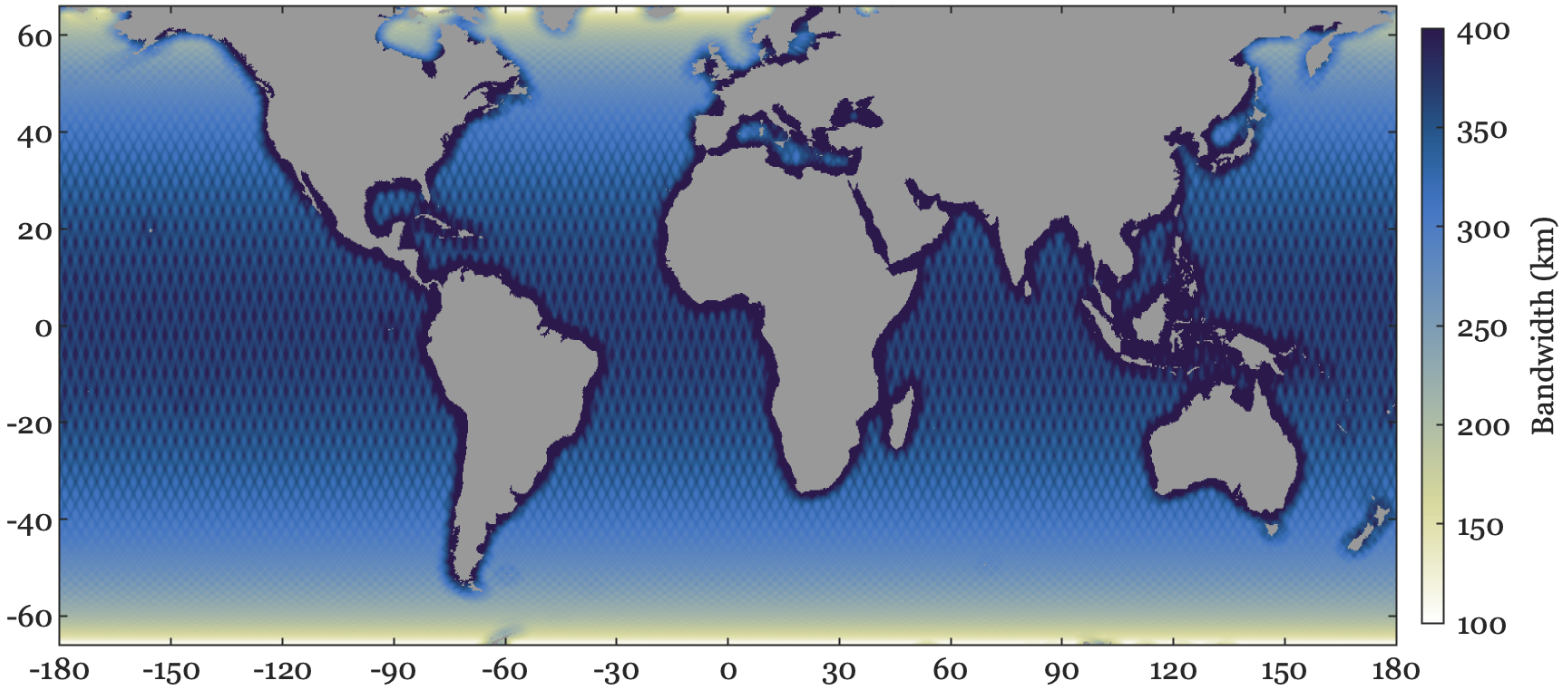
Zonal Mean Weighting Kernels



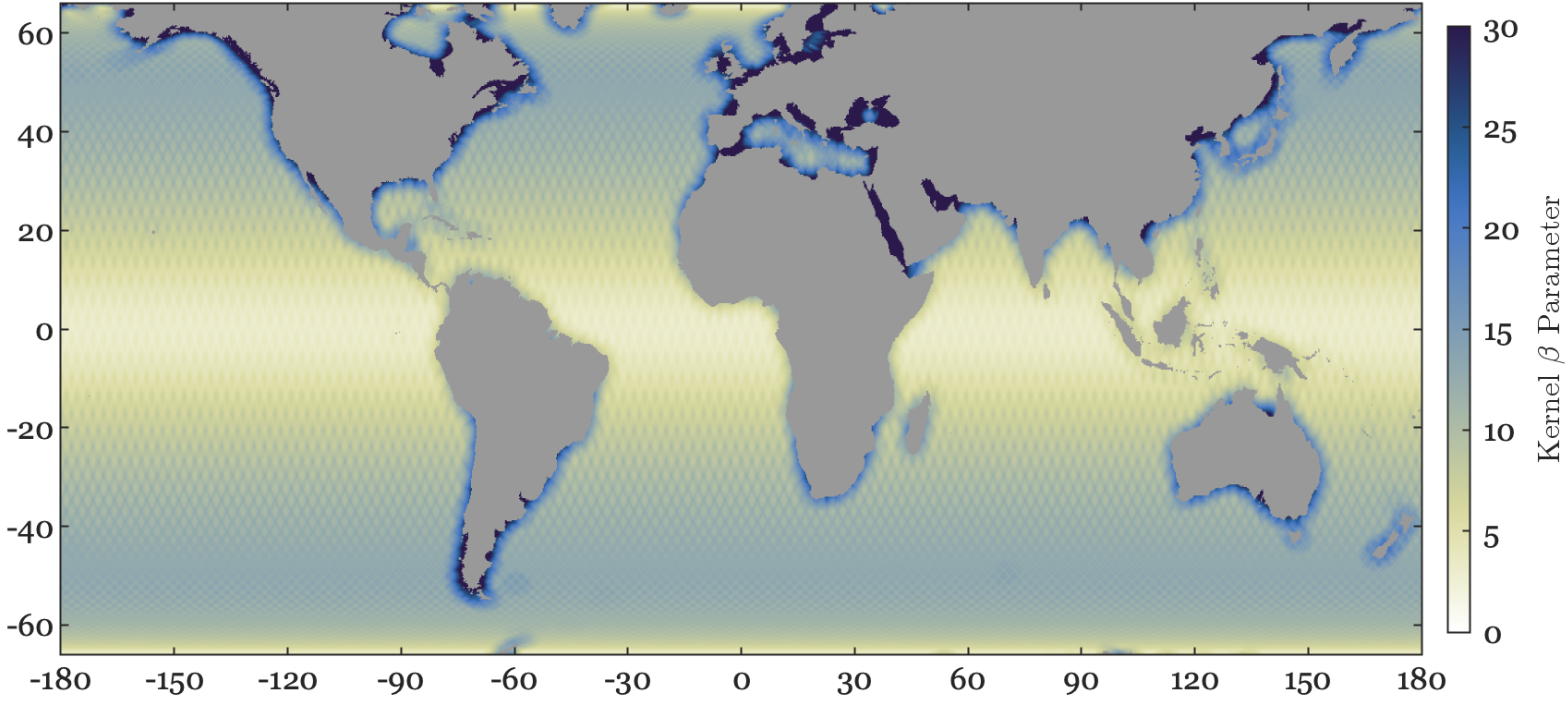
Inner Radius



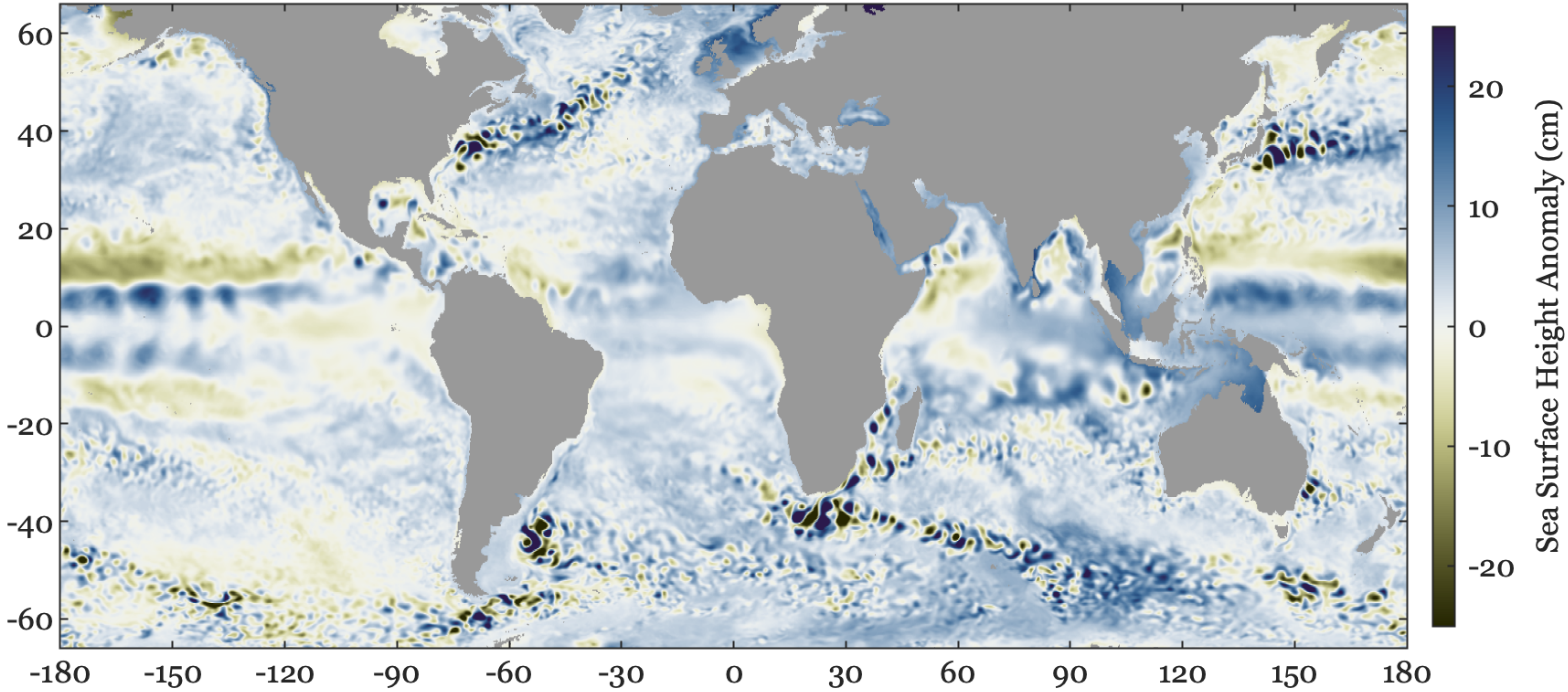
Bandwidth or Outer Radius



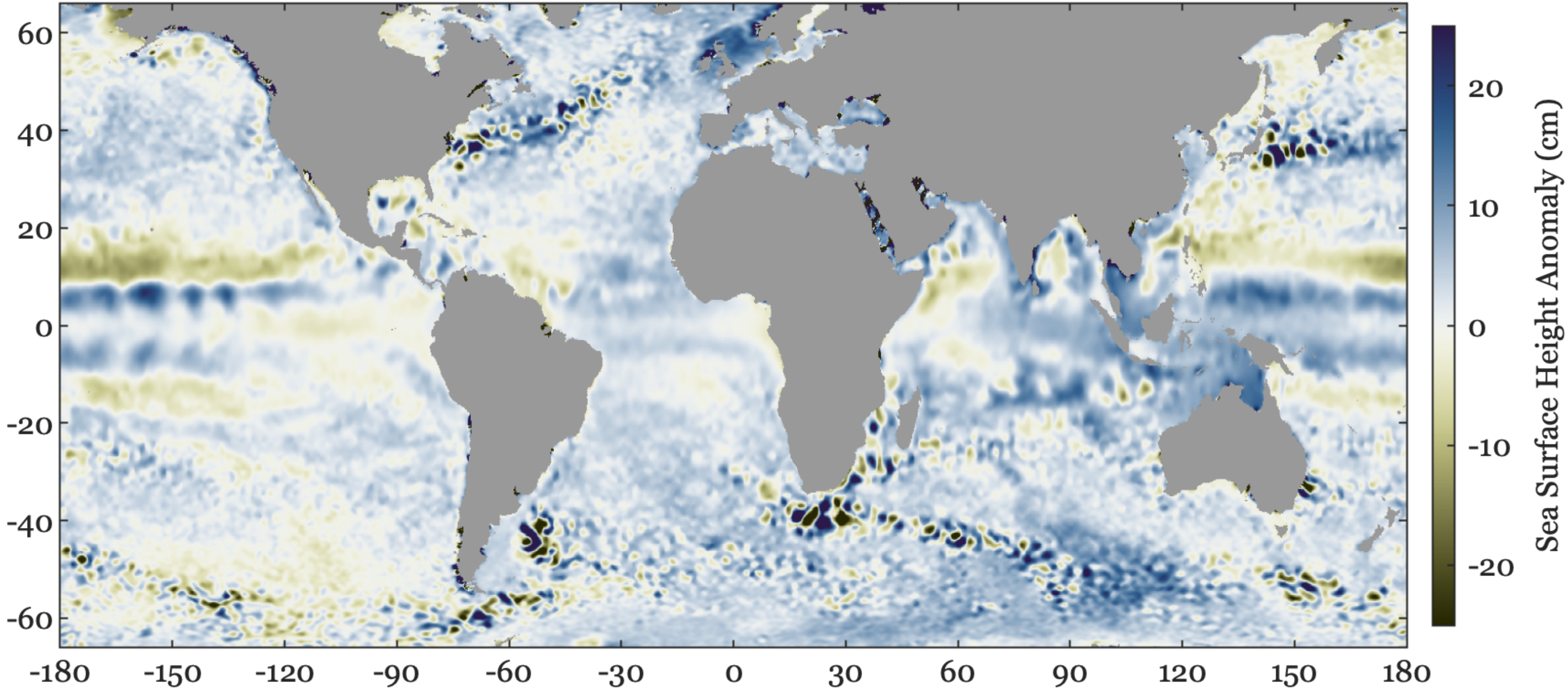
Kernel β Parameter



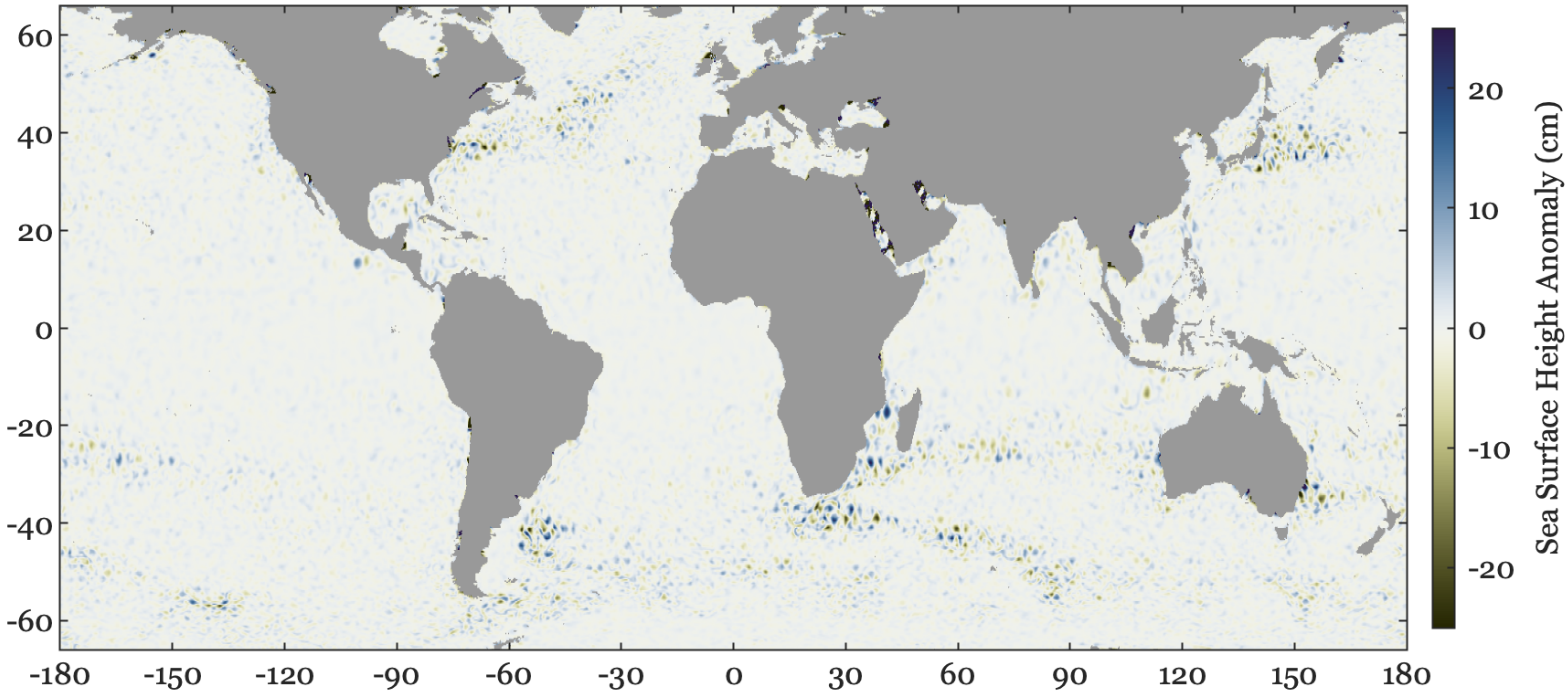
Original Model SSH Anomaly



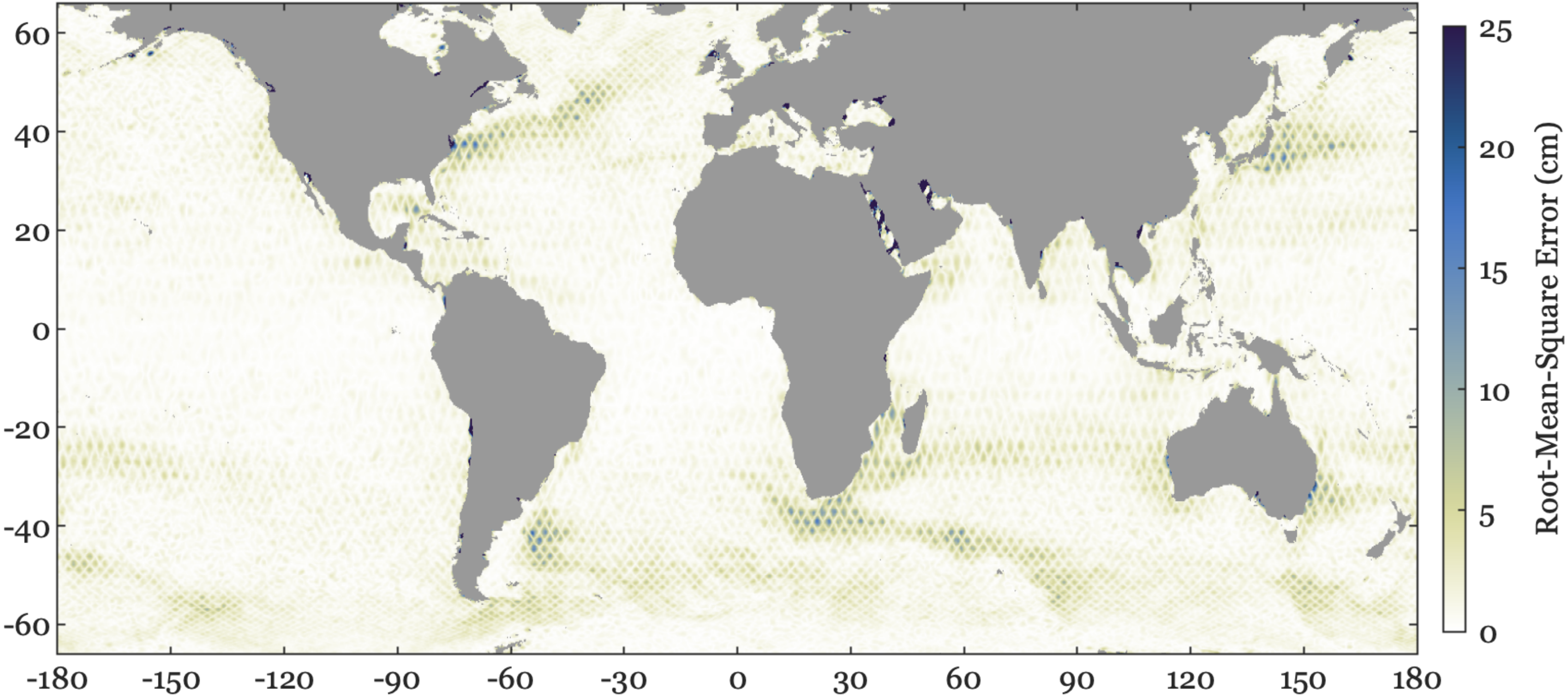
Jason-Reconstructed SSH Anomaly



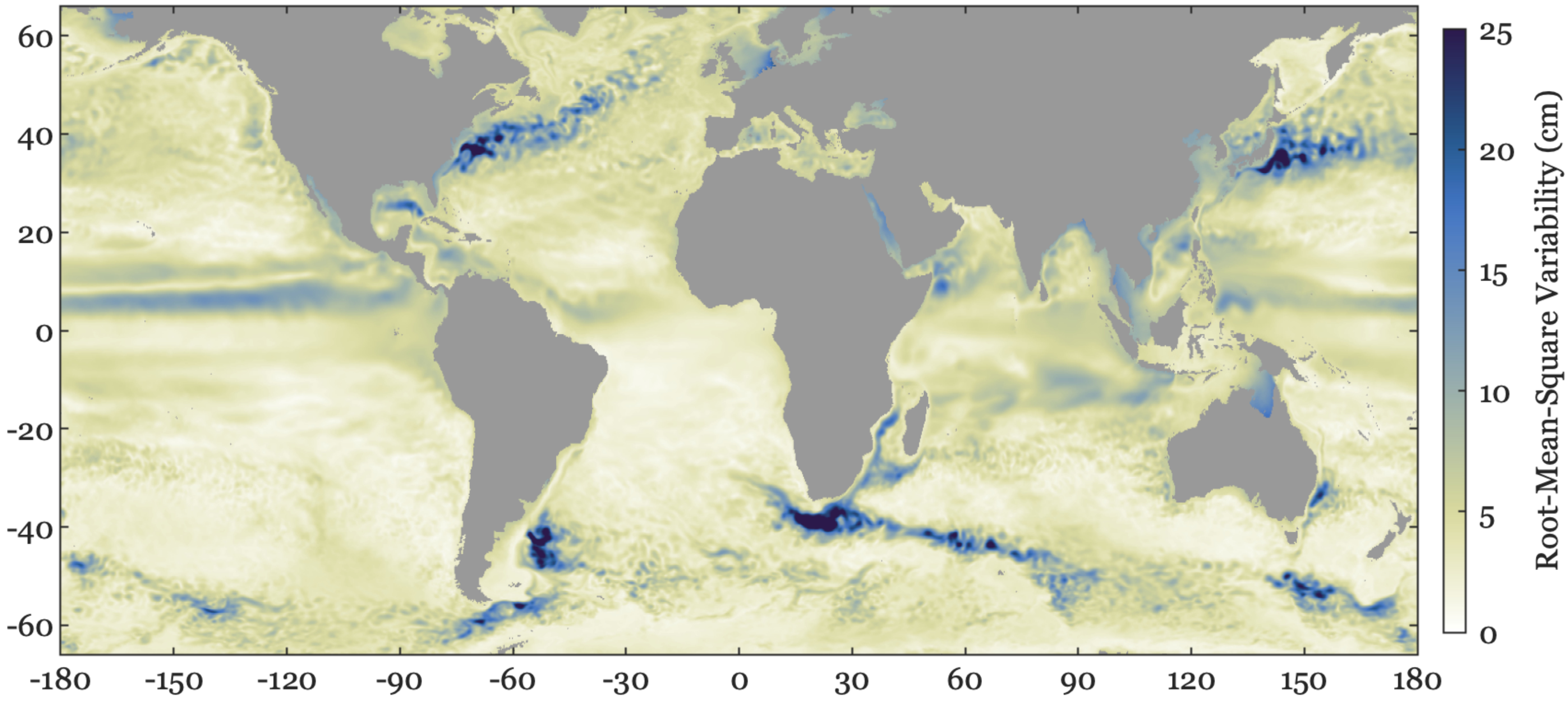
Difference


60
40
20
0
-20
-40
-60
-180
-150
-120
-90
-60
-30
0
30
60
90
120
150
180
20
10
0
-10
-20
Sea Surface Height Anomaly (cm)

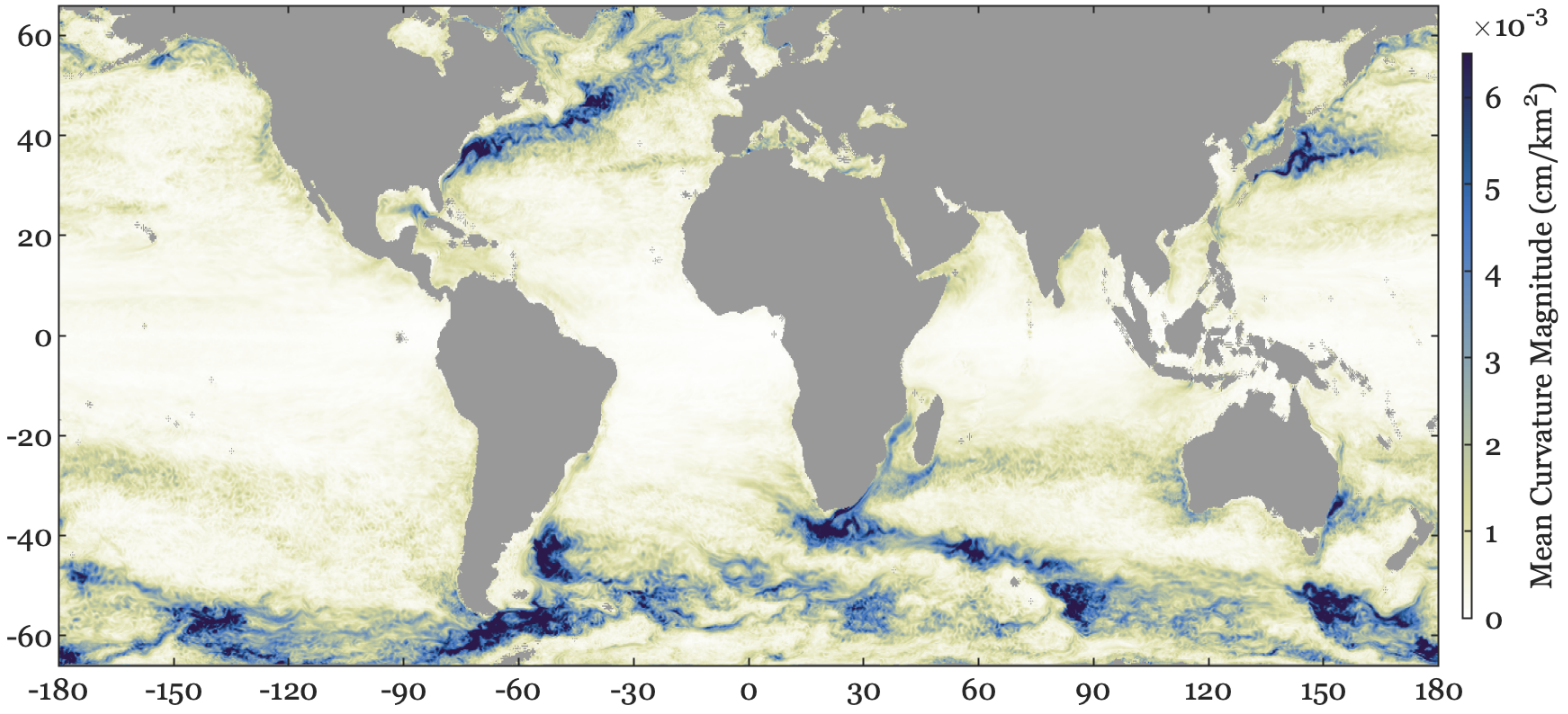
Root-Mean-Square Error



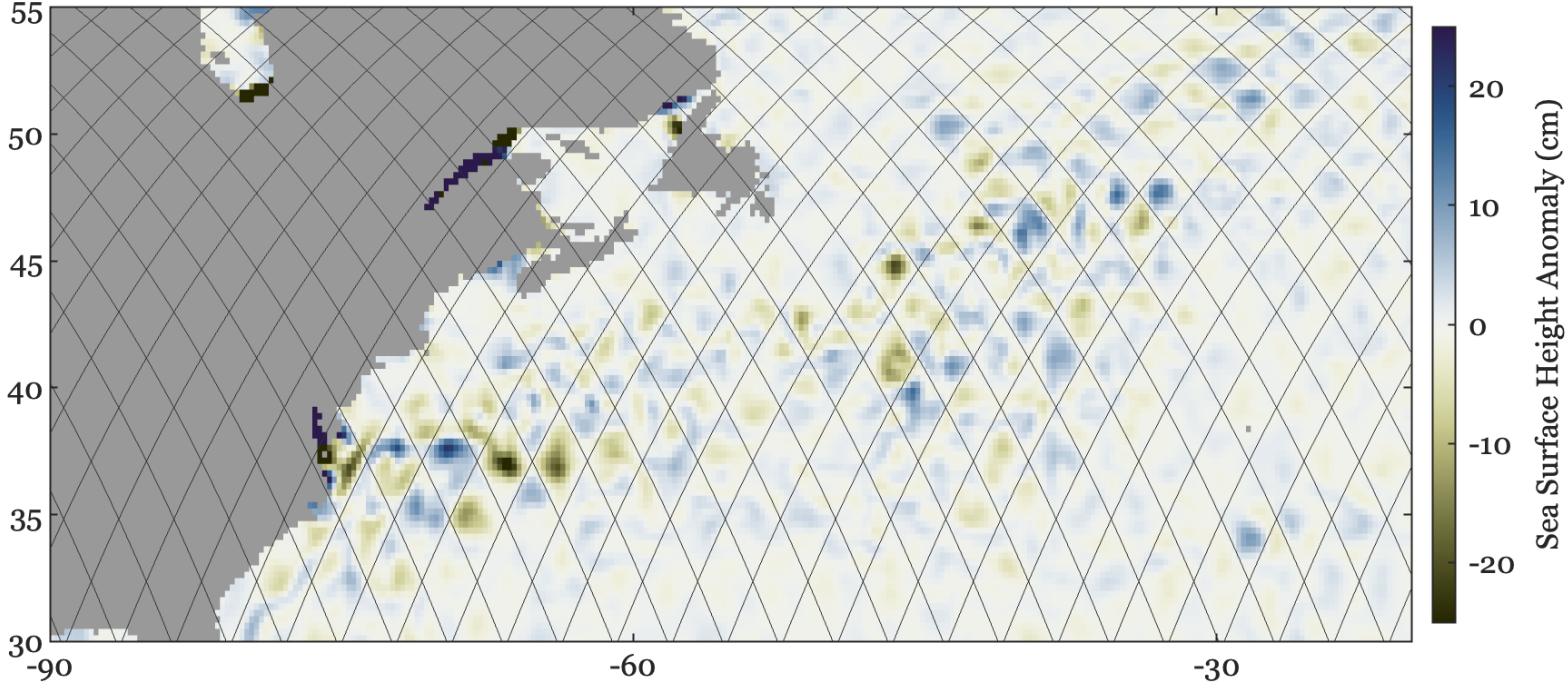
Model Root-Mean-Square Variability



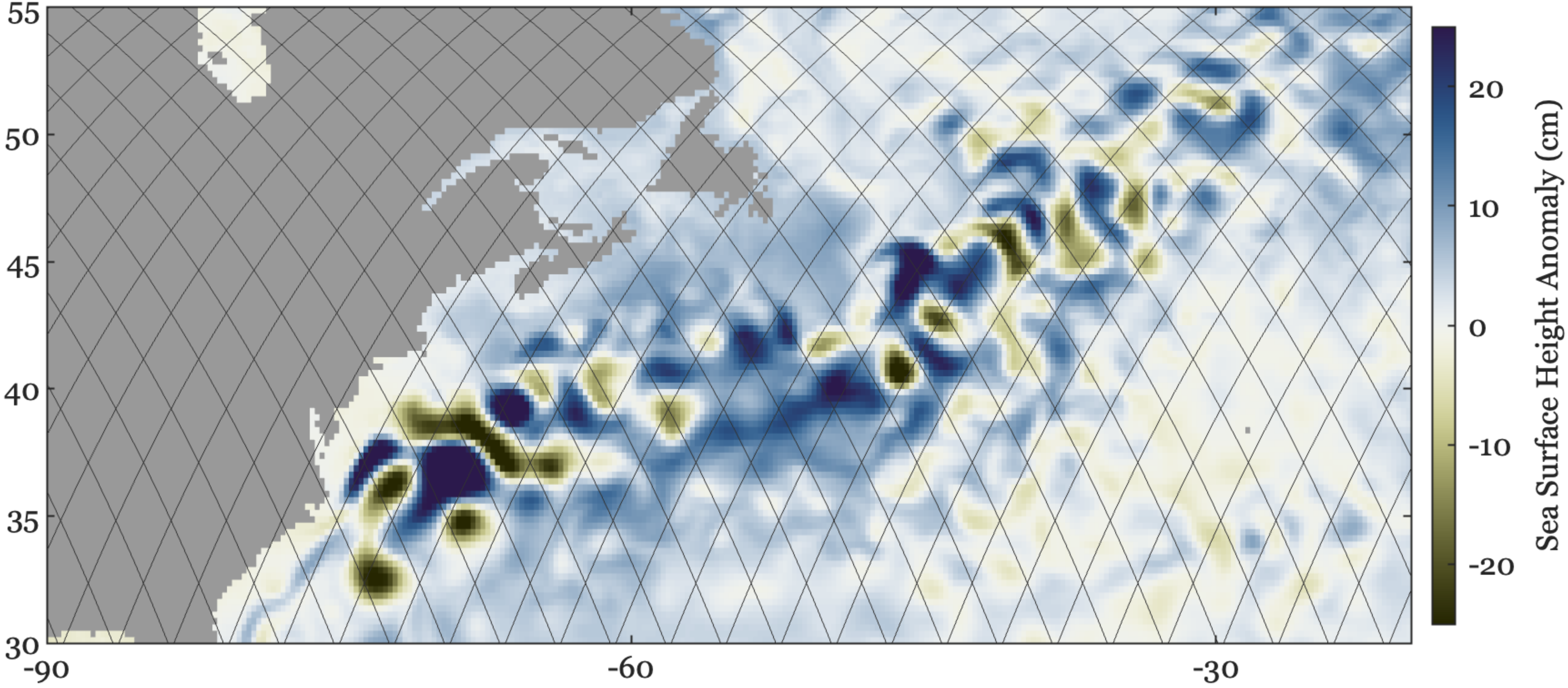
Mean Curvature Magnitude



Difference Close-Up



Model SSH Anomaly Close-Up



Take-Aways

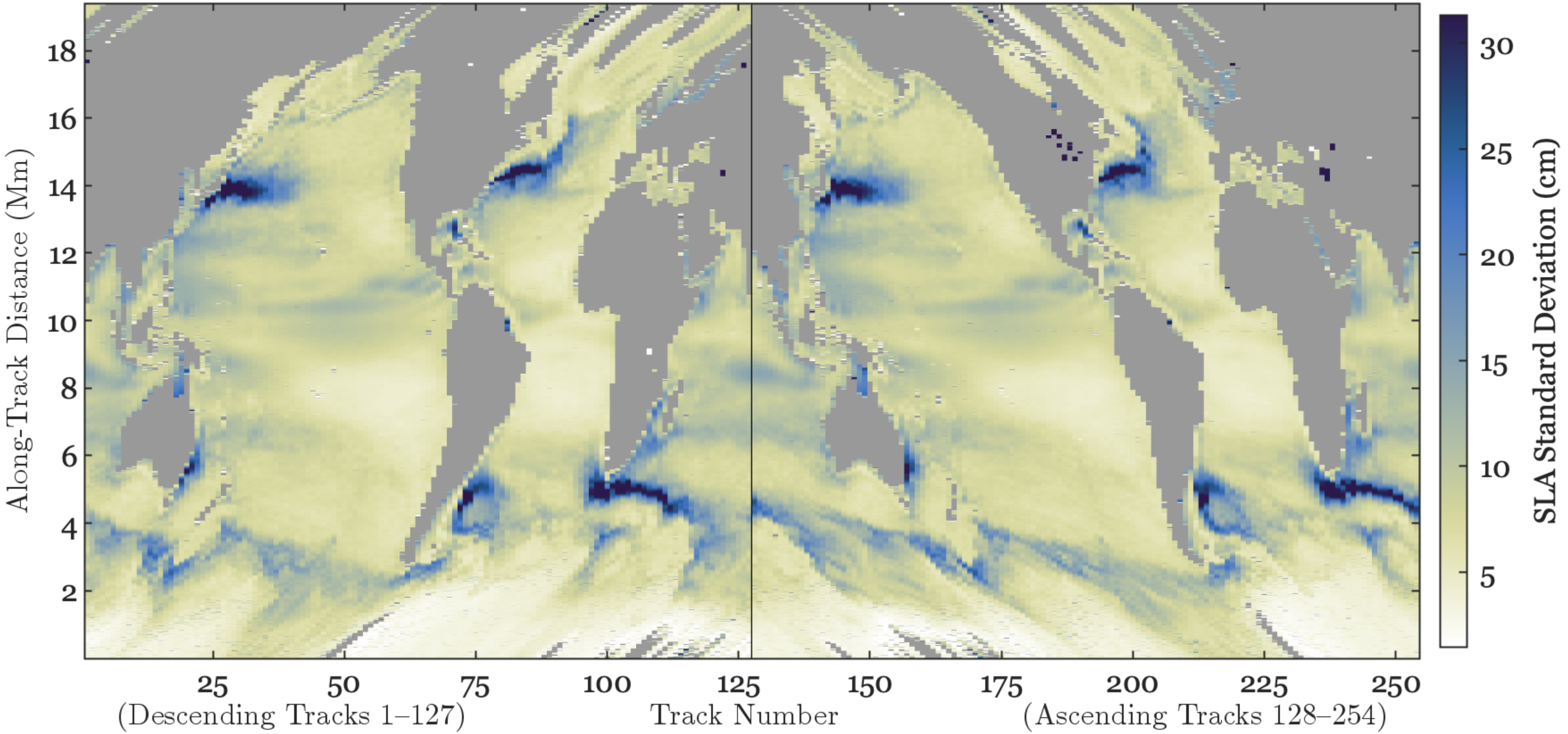
- Local polynomial fitting with linear or quadratic fit, and variable bandwidth, is highly design adaptive, and a promising method for altimetric mapping.
- Perhaps unintuitively, design adaptivity means that spatially varying kernel properties make best use of inhomogeneous data spacing while minimizing artifacts.
- Jason-only maps rivaling current multimission gridded products suggests considerable room for improvements in the latter through mapping refinements.
- The source of error in between-track curvature, together with the equivalence of linear methods, makes it unlikely any method will lead to substantial further improvements.
- This method is particularly promising when dealing with highly heterogenous data, e.g., SWOT + along-track.
- Great potential for an off-the-shelf, user-applied general-purpose mapping algorithm, complementing one-size-fits all black box final products.

Status

- Fully working version of code already available. See "polysmooth" in jLab at www.jmlilly.net.
- Completely factored code with O(10x) speed improvement coming soon.
- Reformatted Jason along-track data w/ noise estimate available at www.jmlilly.net.
- GOLD-derived OSSE test datasets coming soon.
- Generalized beta kernel theory and properties complete.
- Modification for spherical geometry complete.
- Synoptic and asynoptic error theory complete.
- Adaption of asymptotic bandwidth selection theory to the 2D problem complete.
- Writeup in progress!
- Maybe later… application of anisotropic / elliptical kernels to altimetry.

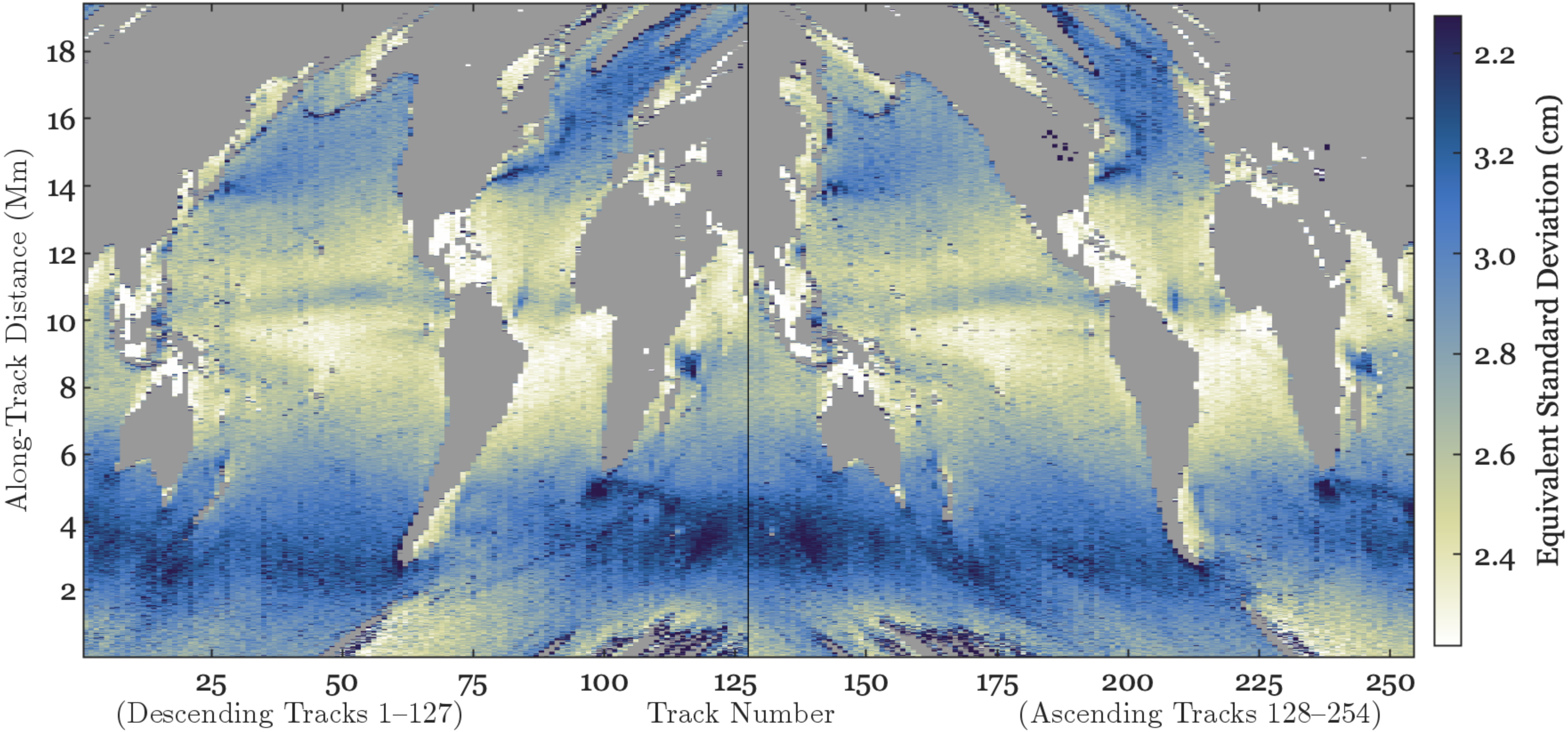
The End. Thanks!!

Jason-Class Along-Track Data

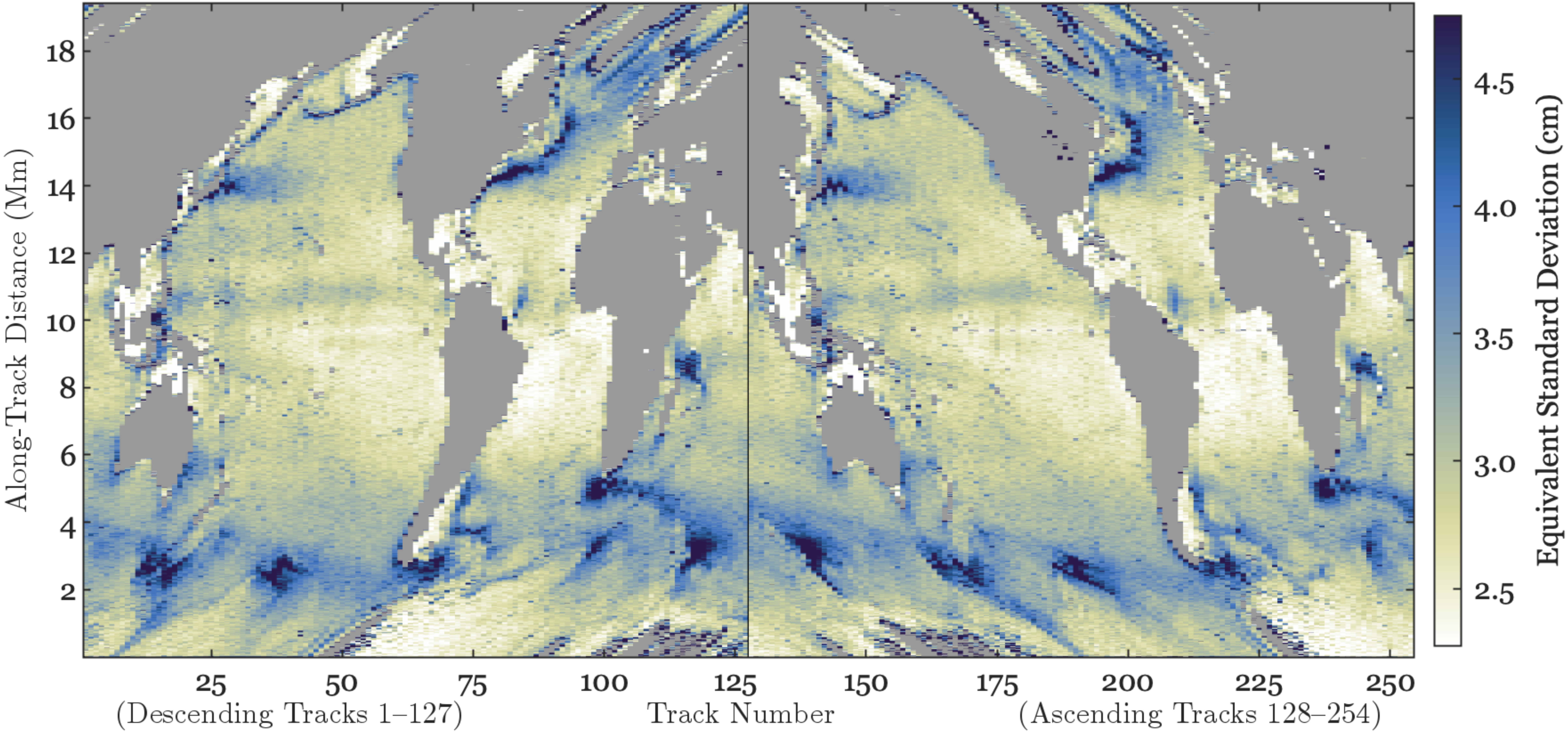


From JasonAlongTrack, a value-added version of the Integrated Multi-Mission Ocean Altimeter Data for Climate Research Version 5.1 by Beckley et al.

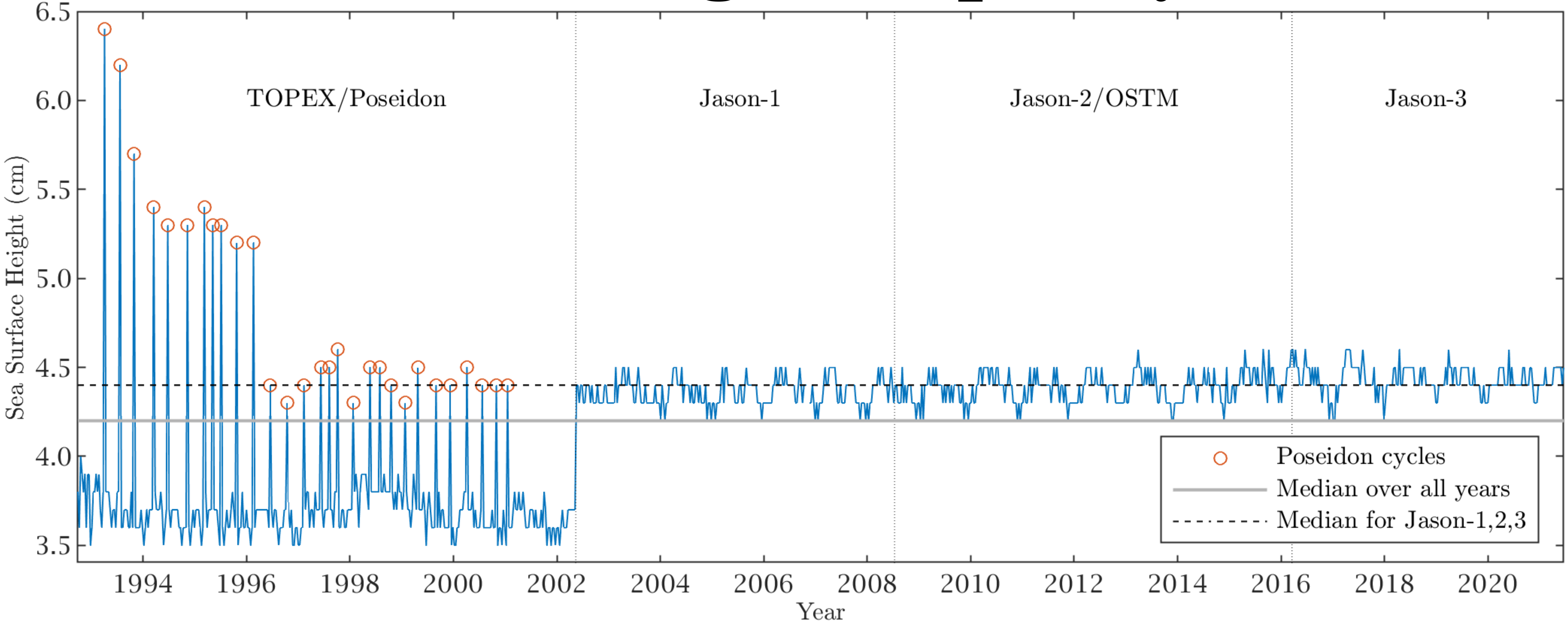
High-Frequency Noise Estimate



Compare with Next-Highest Band



Evolution of High-Frequency Noise



Global median absolute deviation of high-frequency noise estimate.

Little dependence on mission.

The JasonAlongTrack Dataset

We will work with a modified version of the Integrated Multi-Mission Ocean Altimeter Data for Climate Research Version 5.1 by Beckley et al., an improved, integrated, ~30 Jason-class along-track altimeter dataset.

JasonAlongTrack is a reformatted and value-added version of the Beckley et al. product.

Lilly, J. M. (2023). JasonAlongTrack: A reformatted version of the Integrated Multi-Mission Ocean Altimeter Data for Climate Research Version 5.1 (1.0.0) [Data set]. Zenodo. https://doi.org/10.5281/zenodo.10055671